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The baryonic branch of Klebanov-Strassler solution: a supersymmetric family of SU(3) structure backgrounds

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Abstract: We exhibit a one-parameter family of regular supersymmetric solutions of type IIB theory that describes the baryonic branch of the Klebanov-Strassler (KS) theory. The solution is obtained by applying the supersymmetry conditions for SU(3)-structure manifolds to an interpolating ansatz proposed by Papadopoulos and Tseytlin. Other than at the KS point, the family does not have a conformally-Ricci-flat metric, neither it has self-dual three-form flux. By varying also the string coupling, our solution smoothly interpolates between Klebanov-Strassler and Maldacena-Nuñez (MN). The asymptotic IR and UV are that of KS throughout the interpolating flow, except for the extremal value of the parameter where the UV solution drastically changes to MN.

Keywords: Superstring Vacua, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

Supergravity solutions with fluxes are a rich subject that ranges from the physics of string compactifications and vacua to the AdS/CFT correspondence. The geometry of such solutions has recently received much attention. Our improved understanding of this subject reveals that many interesting features are still to be uncovered and that, most likely, new relevant solutions are still to be found. In this paper, using the supersymmetry conditions recently derived in [1], we study a class of solutions relevant for the AdS/CFT correspondence. Up to now only two regular supergravity backgrounds are known that correspond to dual confining $\mathcal{N} = 1$ SYM theories, the Klebanov-Strassler (KS [2]) and the Maldacena-Nuñez (MN [3]) solutions. It was suggested in [4, 5] that the KS solution should belong to a one parameter family of supersymmetric backgrounds describing the baryonic branch of the dual gauge theory. This expectation was strengthen in [5] (Gubser-Herzog-Klebanov, GHK) by a linearized analysis that exhibits a massless Goldstone boson and a first order deformed solution of the equations of motion. In this paper we will show that there exists indeed a one parameter family of regular supersymmetric deformations of the KS solution. The supergravity solution also depends on a second parameter, since we are free to add an arbitrary constant to the dilaton. The two parameters can be interpreted in the dual gauge theory as a baryonic VEV and a gauge coupling constant. By varying both parameters, we find a flow whose endpoint is the MN solution. The result thus makes justice of the many attempts of finding a supergravity solution interpolating between KS and MN [5]–[7].

Supersymmetry in compactifications without fluxes requires the internal manifold to be a Calabi-Yau. The presence of fluxes back-reacts on the geometry transforming the manifold into a generalized Calabi-Yau [1]. In particular, since supersymmetry requires the existence of at least a nonvanishing spinor, the internal manifold must have an SU(3) structure. A set of general conditions that preserve supersymmetry is known for type II theories. In type IIB theories, the internal manifold must be complex [1, 8, 9]. In addition to this, supersymmetry implies a set of linear conditions on the SU(3) torsions and the fluxes [1]. We will apply these supersymmetric constraints to an ansatz for the metric and fluxes proposed by Papadopoulos and Tseytlin (PT [10]), which contains both the KS and MN solutions. We will find that, quite amazingly, the apparently overdetermined set of equations admits a class of solutions. All unknown functions in the ansatz are explicitly determined in terms of two quantities for which we provide a system of coupled first-order differential equations. The MN and KS backgrounds and the linearized deformation of KS found in [5] are particular solutions of these equations. More generally, by a numerical and power series analysis, we can show that there is a one parameter family of regular solutions that deforms KS, while preserving an $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry. We can extend the solution symmetrically with respect to KS, but the two branches are related by a $\mathbb{Z}_2$ symmetry.

From the field theory perspective, the KS solution is known to represent an $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ $\mathcal{N}=1$ gauge theory undergoing a cascading series of Seiberg dualities that decreases the number of colors and terminates with a single $\mathrm{SU}(M)$ gauge group. In the last step of the cascade, the gauge theory is $\mathrm{SU}(2M)\times\mathrm{SU}(M)$ and has a moduli space of vacua that include a baryonic branch. The moduli space is labeled by a complex parameter and has a $\mathbb{Z}_2$ symmetry. It was proposed in [4, 5] that the KS solution should describe the symmetric point on this branch. Our family of solutions, which preserves $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and has a $\mathbb{Z}_2$ symmetry, is the natural candidate for describing the full baryonic branch of the gauge theory. An analysis of the UV asymptotic reveals that the set of operators that are acquiring a VEV is compatible with this interpretation. Moreover, the analysis of the IR behavior of the metric shows that the physics of confinement is the same as described in [2, 3]. In particular, the ratio of string tensions [11, 12] smoothly interpolates between the known cases of KS and MN [13].

The existence of a family of regular solutions describing the moduli space of a gauge theory is particularly significant. All other attempts of deforming a known solution along a moduli space always gave rise to singular backgrounds. For instance, solutions dual to the Coulomb branch of vacua in $\mathcal{N}=4$ or $\mathcal{N}=2$ theories contain singularities corresponding

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to distributions of branes for whose resolution stringy effects are usually invoked [14]. The solution presented here is the first example of gauge theory which can be described in a controllable way along its moduli space. Only for large values of the baryonic VEV the supergravity solution becomes strongly coupled.[1]

Moving along the baryonic branch implies fixing the boundary conditions for the dilaton. By varying also the string coupling, our solution smoothly interpolates between KS and MN. All the backgrounds along this flow have the UV asymptotic behavior of the KS solution, except for the extremal value of the interpolating parameter, where it suddenly changes, the dilaton blows up and we recover the MN solution. Through this paper, we will refer to this solution as the "interpolating flow". From the field theory point of view, we can reach the MN point by varying both the baryonic VEV and a coupling constant.

We have also explicitly checked that the equations of motions are automatically satisfied by the first order system of susy equations we derived. We are therefore sure that the one parameter family of solutions we have found are true solutions of type IIB supergravity.

The paper is organized as follows. In section 2, we review the supersymmetry conditions for type IIB solutions with SU(3) structure. In section 3 we present the PT ansatz and we discuss the choice of complex structure on the manifold. We give the general solution of the supersymmetry conditions and we show how the known cases (MN, KS and GHK) fit in it. The details of the derivation are reported in the Appendices. In section 4 we discuss the family of regular solutions that interpolates between KS and MN. Finally, in section 5, we compare our results with the field theory expectations.

2. Type IIB solutions with SU(3) structure

We are interested in type IIB backgrounds with non-zero RR fluxes and where the space-time is a warped product of the form $\mathbb{R}^{1,3} \times_w M_6$

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + ds_6^2\,, \tag{2.1}$$

where A is a function of the internal coordinates.

For these backgrounds a set of general conditions on the six-dimensional manifold and the fluxes to preserve supersymmetry is known [1, 8, 9]. In particular supersymmetry forces the manifold M_6 to have a globally defined invariant spinor. This is possible only for manifolds that have reduced structure. The structure group of a manifold is the group of transformations required to patch the orthonormal frame bundle. A six dimensional riemannian manifold has automatically SO(6) structure. In order to preserve supersymmetry, the structure group of M_6 should be a subgroup of SO(6). In general, the smaller the group, the bigger the number of supercharges of the effective four dimensional theory. To preserve minimal supersymmetry ($\mathcal{N} = 1$ in four dimensions) the structure group should be at most SU(3). The set of supersymmetry conditions for IIB backgrounds with SU(3)-structure was derived in [1]. We refer to this paper for the detailed derivation, here we briefly summarize the results. For details about SU(3) structure (or G-structures in general) we refer to [15, 16, 17] and references therein.

[1]We thank Igor Klebanov and Anatoly Dymarsky for a useful discussion about this issue.

	$\mathcal{W}$	NS	RR
$1 \oplus \bar{1}$	W_1	$H^{(1)}$	$F_3^{(1)}$
$3 \oplus \bar{3}$	W_4, W_5	$H^{(3)}$	$F_1^{(3)}$, $F_3^{(3)}$, $F_5^{(3)}$
$6 \oplus \bar{6}$	W_3	$H^{(6)}$	$F_3^{(6)}$
$8 \oplus 8$	W_2	$-$	$-$

Table 1: Decomposition of torsion and fluxes in SU(3) representations.

A manifold with SU(3) structure has all the group-theoretic features of a Calabi-Yau, namely an invariant spinor η and two- and three forms, J and Ω respectively, that are constructed as bilinears of the spinor. If the manifold had SU(3) *holonomy*, not only J and Ω would be well defined, but also they would be closed: $dJ = 0 = d\Omega$. If they are not, dJ and $d\Omega$ give a good measure of how far the manifold is from having SU(3) holonomy. The failure of an SU(3) structure to become SU(3) holonomy is measured by the intrinsic torsion. In terms of SU(3) representations, the intrinsic torsion has $(3 \oplus \bar{3} \oplus 1) \otimes (3 \oplus \bar{3})$ components which are defined as follows:

$$
\begin{aligned}
dJ &= -\frac{3}{2}\,\mathrm{Im}(W_1\bar{\Omega}) + W_4 \wedge J + W_3\,, \\
d\Omega &= W_1 J^2 + W_2 \wedge J + W_5^{(\bar{3})} \wedge \Omega\,.
\end{aligned} \tag{2.2}
$$

W_1 is a complex zero-form in $1 \oplus 1$, W_2 is a complex primitive two-form, so it lies in $8 \oplus 8$, W_3 is a real primitive $(2,1) \oplus (1,2)$ form in the $6 \oplus \bar{6}$, W_4 is a real one-form in $3 \oplus \bar{3}$[2] and finally $W_5^{(\bar{3})}$ is a complex $(0,1)$-form (notice that in (2.2) the $(1,0)$ part drops out), so its degrees of freedom are again $3 \oplus \bar{3}$.

$W_1 = W_2 = 0$ corresponds to an integrable complex structure (dJ does not have (3,0) or (0,3) pieces, and $d\Omega$ misses the (2,2) pieces), which should be the case for a complex manifold. A conformal Calabi-Yau (a space with a conformally-Ricci flat metric) has $W_1 = W_2 = W_3 = 0$, and non-zero W_4 and W_5 obeying $3W_4 = 2W_5$.

Similarly all the fluxes in the theory can be decomposed in SU(3) representations

$$
\begin{aligned}
H &= -\frac{3}{2}\,\mathrm{Im}\left(H^{(1)}\bar{\Omega}\right) + H^{(3+\bar{3})} \wedge J + H^{(6+\bar{6})}\,, \\
F_3 &= -\frac{3}{2}\,\mathrm{Im}\left(F_3^{(1)}\bar{\Omega}\right) + F_3^{(3+\bar{3})} \wedge J + F_3^{(6+\bar{6})}\,.
\end{aligned} \tag{2.3}
$$

Using group theory it is then possible to reduce the supersymmetry conditions on the variations of the fermions to a set of algebraic equations for the different SU(3) components of the torsion and the fluxes

Since torsion is induced by the fluxes in order to have a supersymmetric solution non-zero torsions must be compensated by non-zero fluxes in the same representation.

Table 1 shows that in IIB theories there are no fluxes to compensate torsion in the $8 \oplus 8$. Thus $W_2 = 0$ in any IIB solution with SU(3) structure.

[2]In the following we shall decompose the torsions W_3, W_4 and W_5 in the complex basis as $W_3 = W_3^{(6)} + W_3^{(\bar{6})}$, $W_4 = W_4^{(3)} + W_4^{(\bar{3})}$ and $W_5 = W_5^{(3)} + W_5^{(\bar{3})}$.

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Supersymmetry imposes additionally that all the singlets ($W_1, F_3^{(1)}$ and $H^{(1)}$) vanish

$$W_1 = F_3^{(1)} = H^{(1)} = 0\,. \tag{2.4}$$

Therefore we get the first basic fact about IIB backgrounds with SU(3) structure - they necessarily involve complex manifolds [1, 8, 9]:

$$W_1 = W_2 = 0\,. \tag{2.5}$$

In order to write down the conditions for the $3 \oplus \bar{3}$ and $6 \oplus \bar{6}$ components we need to consider the decomposition of the ten-dimensional supersymmetry parameter in terms of the four-dimensional one and the SU(3)-invariant spinor. The two ten-dimensional spinors have the same chirality and decompose as $\epsilon_i = \zeta_+ \otimes \eta_+^i + \zeta_- \otimes \eta_-^i$, where $i = 1, 2$ and $\zeta_+^* = \zeta_-$, $\eta_+^{i*} = \eta_-^i$. The six-dimensional spinors $\eta_\pm^i$ are related to the SU(3) invariant spinor η_+ by the functions α and β: $\eta_+^1 = \frac{1}{2}(\alpha + \beta)\eta_+$ and $\eta_+^2 = \frac{1}{2i}(\alpha - \beta)\eta_+$, where η_+ is normalized as $\eta_+^\dagger \eta_+ = \frac{1}{2}$.

For $6 \oplus \bar{6}$ one gets the following equations for W_3, $F_3^{(6)}$ and $H^{(6)}$

$$\begin{aligned}
(\alpha^2 - \beta^2)\, W_3^{(6)} &= e^\phi\, 2\alpha\beta\, F_3^{(6)}\,, \\
(\alpha^2 + \beta^2)\, W_3^{(6)} &= -2\alpha\beta\, *_6\, H^{(6)}\,, \\
(\alpha^2 - \beta^2)\, H^{(6)} &= e^\phi\, (\alpha^2 + \beta^2)\, *_6\, F_3^{(6)}\,.
\end{aligned}$$

The last equation is related to the self-duality of the complex 3-form flux $G_3 = F_3 - ie^{-\phi}H$. In our conventions a primitive (1,2) form — which transforms in the 6 representation — is imaginary anti self-dual (AISD), while a primitive (2,1) form — transforming in the $\bar{6}$ — is ISD.

The analysis for the components $3 \oplus \bar{3}$ is more involved and depends on the choice of the functions α and β. The values $\alpha = 0$ or $\beta = 0$; $\alpha = \pm\beta$ and $\alpha = \pm i\beta$ are special, since there W_3 or $F_3^{(6)}$ or $H^{(6)}$ vanish. These correspond to well-known cases, which have been labeled respectively type B, A and C solutions. The full set of conditions for the A [20], B [18, 19] and C [7] solutions in a compact form are summarized in table 2 (quantities not mentioned in the table in a given representation are vanishing).

Type B and C include as examples the two known supergravity solutions dual to $\mathcal{N} = 1$ Super Yang-Mills, the KS and MN solutions.

The KS solution describes the near horizon geometry of N regular and M fractional D3 branes at the tip of the conifold, these last being D5 branes wrapping a collapsed 2-cycle of the conifold. The geometry is the deformed conifold with a conformal factor (a "throat") due to the presence of fluxes. The topology is $\mathbb{R} \times S^2 \times S^3$, where at the apex of the cone the S^2 shrinks to zero size while the S^3 remains finite.

The solution falls into the B class of table 2 for $\beta = 0$. The dilaton is constant. The metric is conformally Ricci-flat, satisfying $W_1 = W_2 = W_3 = 0$ and $3W_4 = 2W_5$. The fluxes H and F_3 have only non-vanishing components in the $6 + \bar{6}$, and satisfy $e^\phi * F_3 = H$. Thus the standard NS-RR three-form combination $G_3 = F_3 - ie^{-\phi}H$ is imaginary-self-dual (ISD).

<table>
<tr><th></th><th>$\alpha = \pm\beta$ (A)</th><th>$\alpha = 0$ or $\beta = 0$ (B)</th><th>$\alpha = \pm i\beta$ (C)</th></tr>
<tr><td>1</td><td colspan="3">$W_1 = F_3^{(1)} = H^{(1)} = 0$</td></tr>
<tr><td>8</td><td colspan="3">$W_2 = 0$</td></tr>
<tr><td>6</td><td>$F_3^{(6)} = 0$
$W_3 = \mp * H^{(6)}$</td><td>$W_3 = 0$
$\mp e^{\phi} * F_3^{(6)} = H^{(6)}$</td><td>$H^{(6)} = 0$
$W_3 = \pm e^{\phi} * F_3^{(6)}$</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">$W_5^{(\bar{3})} = 2W_4^{(\bar{3})} =$
$\pm 2iH^{(\bar{3})} = 2\bar{\partial}\phi$
$\bar{\partial}A = \bar{\partial}\alpha = 0$</td><td>$e^{\phi}F_5^{(\bar{3})} = \frac{2}{3}iW_5^{(\bar{3})} =$
$iW_4^{(\bar{3})} = -2i\bar{\partial}A =$
$-4i\bar{\partial}\log\alpha$
$\bar{\partial}\phi = 0$</td><td rowspan="2">$\pm e^{\phi}F_3^{(\bar{3})} = 2iW_5^{(\bar{3})} =$
$-2i\bar{\partial}A = -4i\bar{\partial}\log\alpha =$
$-i\bar{\partial}\phi$</td></tr>
<tr><td>$e^{\phi}F_1^{(\bar{3})} = 2e^{\phi}F_5^{(\bar{3})} =$
$iW_5^{(\bar{3})} = iW_4^{(\bar{3})} = i\bar{\partial}\phi$</td></tr>
</table>

Table 2: Special IIB solutions with SU(3) structure.

The MN solution describes N D5 branes wrapping the S^2 in a six-dimensional manifold with topology $\mathbb{R} \times S^2 \times S^3$. It is a type C solution with $\beta = i\alpha$. The only non trivial flux is the RR three form which has components in the $6 + \bar{6}$ and in the $3 + \bar{3}$. The first two components of the intrinsic torsion are zero, $W_1 = W_2 = 0$. The manifold is then complex but not conformal Calabi-Yau since W_4 is also zero while W_5 is not. W_5 is related to the dilaton, the vector component of the RR flux and the warp factor as in Table 2. W_3 matches the RR flux in the 6, namely $W_3^{(6)} = e^{\phi} * F_3^{(6)}$.

Away from the three special cases, a general set of equations for $3 \oplus \bar{3}$ can be written, but these depend on the phases of α and β. It can be shown that for all Type IIB solutions $|\alpha|^2 + |\beta|^2 = e^A$. Furthermore, we can change α and β by the same phase with a rotation of the spinor η_+. A possible choice to describe type B, C and interpolating BC solutions is to take α real and β imaginary. Using this gauge fixing, the resulting set of equations is

$$\begin{aligned}
e^{\phi}F_3^{(\bar{3})} &= \frac{8\alpha\beta(\alpha^2-\beta^2)}{\alpha^4-6\alpha^2\beta^2-3\beta^4}\bar{\partial}\log\alpha\,,\\
e^{\phi}F_5^{(\bar{3})} &= \frac{-4i(\alpha^2+\beta^2)(\alpha^2-\beta^2)}{\alpha^4-6\alpha^2\beta^2-3\beta^4}\bar{\partial}\log\alpha\,,\\
H^{(\bar{3})} &= \frac{8i\alpha\beta(\alpha^2+\beta^2)}{\alpha^4-6\alpha^2\beta^2-3\beta^4}\bar{\partial}\log\alpha\,,
\end{aligned} \tag{2.6}$$

for the fluxes, and

$$\begin{aligned}
W_4^{(\bar{3})} &= \frac{-4(\alpha^2+\beta^2)^2}{\alpha^4-6\alpha^2\beta^2-3\beta^4}\bar{\partial}\log\alpha\,, & \bar{\partial}\phi &= \frac{-16\alpha^2\beta^2}{\alpha^4-6\alpha^2\beta^2-3\beta^4}\bar{\partial}\log\alpha\,,\\
W_5^{(\bar{3})} &= \frac{-2(3\alpha^4+2\alpha^2\beta^2+3\beta^4)}{\alpha^4-6\alpha^2\beta^2-3\beta^4}\bar{\partial}\log\alpha\,, & \bar{\partial}\beta &= \frac{-\beta(3\alpha^4+6\alpha^2\beta^2-\beta^4)}{\alpha^4-6\alpha^2\beta^2-3\beta^4}\bar{\partial}\log\alpha\\
\bar{\partial}A &= \frac{2(\alpha^2-\beta^2)^2}{\alpha^4-6\alpha^2\beta^2-3\beta^4}\bar{\partial}\log\alpha\,,
\end{aligned} \tag{2.7}$$

for the geometry, where, as one can see, all non-vanishing vector components of the torsion and the fluxes are proportional to $\bar{\partial}\log\alpha$. Notice that only for the three special cases the ratio $\bar{\partial}A/\bar{\partial}\log\alpha$ is constant.

This set of equations seems to suggest the existence of a family of supersymmetric type IIB solutions with SU(3) structure interpolating between KS and MN. Finding such a family of backgrounds is the purpose of this paper. At the level of geometry, this involves a family of complex metrics that includes the deformed conifold but is not Ricci-flat (is not Kähler) in general.[3]

3. Supersymmetry conditions for the interpolating solution

We are looking for a family of solutions to the supersymmetry constraints (2.4)–(2.7) that interpolates between the KS and the MN solutions. In [10] Papadopoulos and Tseytlin (PT) proposed an ansatz for IIB solutions with fluxes involving a generalization of the conifold metric [22]. It contains KS and MN as special cases and it also describes the singular and resolved conifold metrics. This ansatz covers hence solutions in the whole of A-B-C triangle. Indeed, as we have already discussed, KS and MN are special cases of type B and C respectively, and by relabeling the fluxes and scaling the metric (S-duality), one can get the A type from C.

The idea is then to use PT ansatz to solve the SU(3) structure equations given in the previous section. The topology of the space described by the ansatz is $\mathbb{R}^{1,3}\times\mathbb{R}\times S^2\times S^3$, where the 6-dimensional metric has SU(2) × SU(2) symmetry. The metric in string frame is given by[4]

$$\begin{aligned} ds^2 &= e^{2A}dx_\mu dx^\mu + e^{-6p-x}dt^2 + ds_5^2 \\ &= e^{2A}dx_\mu dx^\mu + \sum_i^6 E_i^2\,, \end{aligned} \tag{3.1}$$

where p, x and A are functions of the radial coordinate t only. We found it more convenient to use slightly different conventions than PT: our radial variable t is related to the PT one by $du = e^{-4p}dt$ and similarly the function $2A$ corresponds in PT to $2p - x + 2A$. We also define a new set of vielbeins E_i which are more suitable for the SU(3) structure we will introduce later. They are related to the more conventional vielbeins in PT by

$$\begin{aligned} E_1 &= e^{\frac{x+g}{2}}e_1 = e^{\frac{x+g}{2}}d\theta_1\,, \\ E_2 &= e^{\frac{x+g}{2}}e_2 = -e^{\frac{x+g}{2}}\sin\theta_1 d\phi_1\,, \\ E_3 &= e^{\frac{x-g}{2}}\tilde{\epsilon}_1 = e^{\frac{x-g}{2}}(\epsilon_1 - a(t)e_1)\,, \end{aligned}$$

[3] Using the fact that by scaling the fermion one can shift away W_5 one could relate this to the Hitchin's variational problem [21] — indeed what we have here is an example of a family of closed 3-forms and the CY metric should come out of the minimization of the volume functional. In a way one could speculate that including the three-forms H and F_3 one could write a "generalized" Hitchin functional whose variation should be equivalent to IIB equations of motion, and thus our family of backgrounds corresponds to a minimum of a volume functional with fluxes.

[4] PT use Einstein frame metric in [10], but we found it more convenient to use string frame metric

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$$
\begin{aligned}
E_4 &= e^{\frac{x-g}{2}}\tilde{\epsilon}_2 = e^{\frac{x-g}{2}}(\epsilon_2 - a(t)e_2)\,,\\
E_5 &= e^{-3p-\frac{x}{2}}dt = e^{p-\frac{x}{2}}du\,,\\
E_6 &= e^{-3p-\frac{x}{2}}\tilde{\epsilon}_3 = e^{-3p-\frac{x}{2}}(\epsilon_3 + \cos\theta_1 d\phi_1)\,,
\end{aligned}
\tag{3.2}
$$

where g and a are also functions of the radial coordinate only. The S^2 in the metric is parameterized by the coordinates θ_1, ϕ_1 and corresponds to the vielbeins e_1, e_2 (their expression can be read off the definition of E_1 and E_2). Similarly $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ are the left-invariant forms on S^3 with Euler angle coordinates ψ, θ_2, ϕ_2

$$
\begin{aligned}
\epsilon_1 &\equiv \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2\ ,\\
\epsilon_2 &\equiv \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2\ ,\\
\epsilon_3 &\equiv d\psi + \cos\theta_2 d\phi_2\ ,\\
d\epsilon_i &= -\frac{1}{2}\epsilon_{ijk}\epsilon_j\wedge\epsilon_k\ .
\end{aligned}
\tag{3.3}
$$

The fluxes of the PT ansatz respect the SU(2) $\times$ SU(2) symmetry. They are more readable in the original vielbeins e_i and ϵ_l

$$
\begin{aligned}
H &= h_2\,\tilde{\epsilon}_3\wedge(\epsilon_1\wedge e_1 + \epsilon_2\wedge e_2) + \\
&\quad + dt\wedge\left[h_1'(\epsilon_1\wedge\epsilon_2 + e_1\wedge e_2) + \chi'\,(e_1\wedge e_2 - \epsilon_1\wedge\epsilon_2) + h_2'\,(\epsilon_1\wedge e_2 - \epsilon_2\wedge e_1)\right], \quad (3.4)\\
F_3 &= P\left[\tilde{\epsilon}_3\wedge(\epsilon_1\wedge\epsilon_2 + e_1\wedge e_2 - b(\epsilon_1\wedge e_2 - \epsilon_2\wedge e_1)) + b'\,dt\wedge(\epsilon_1\wedge e_1 + \epsilon_2\wedge e_2)\right], \quad (3.5)\\
F_5 &= \mathcal{F}_5 + *\mathcal{F}_5\ , \quad (3.6)\\
\mathcal{F}_5 &= K\,e_1\wedge e_2\wedge\epsilon_1\wedge\epsilon_2\wedge\epsilon_3\,. \quad (3.7)
\end{aligned}
$$

where h_1, h_2, b, χ and K are function of the coordinate t, and primes always denote derivatives with respect to t. The function K is related to h_1, h_2 and b by $K(t) = Q + 2P[h_1(t) + b(t)h_2(t)]$, where the constants Q and P are proportional to the number of regular and fractional D3 branes respectively. In particular, $P = -M\alpha'/4$.

The fluxes and the expression for K are chosen in such a way that they automatically satisfy the Bianchi identities. This implies that the PT ansatz does not allow for the introduction of brane sources and thus always describes the "flux side" of the large N transition [23].

In order to apply the analysis of the previous section to the PT ansatz we have to choose an SU(3) structure, which amounts to choosing a fundamental form and a holomorphic 3-form.

$$
\begin{aligned}
J &= E_1\wedge(\mathcal{A}E_2 + \mathcal{B}E_4) + E_3\wedge(\mathcal{B}E_2 - \mathcal{A}E_4) + E_5\wedge E_6, \quad (3.8)\\
\Omega &= [E_1 + i(\mathcal{A}E_2 + \mathcal{B}E_4)]\wedge[E_3 + i(\mathcal{B}E_2 - \mathcal{A}E_4)]\wedge[E_5 + iE_6]\,. \quad (3.9)
\end{aligned}
$$

$\mathcal{A}$ and $\mathcal{B}$ are functions of the radial direction and correspond to a rotation in the E_2 - E_4 plane (therefore they obey $\mathcal{A}^2 + \mathcal{B}^2 = 1$). The introduction of $\mathcal{A}$ and $\mathcal{B}$ is motivated by the fact that we need a structure interpolating between MN and KS.

This is the structure we will use to decompose the fluxes into SU(3) representations. For the components of the torsions and the fluxes in this complex structure, see the appendix A.

The PT ansatz and the choice of structure involve the following set of unknown functions: A, a , p, x, g (from the metric), h_1, h_2, χ, b (from the fluxes), $\mathcal{A}$, $\mathcal{B}$, α and β (from the SU(3) structure). The strategy is now to plug the ansatz in equations (2.4)–(2.7) and solve the system of differential equations for our functions.

3.1 The supersymmetry conditions

The system of equations for the unknown functions derived from (2.4)-(2.7) seems overdetermined and discouraging. However a lot of patience and a more careful analysis reveals that the number of independent equations matches the number of unknowns. We give here the solution referring to appendix B for a detailed derivation. For simplicity, we remove all integration constants that can be eliminated with a change of coordinates and those that correspond to singular backgrounds. The general solution is given in appendix B.

In what follows we also rescale $P = -1/2$. Inspired by the discussion in Section 2, we choose α real and β purely imaginary. We can parameterize them as follows

$$\beta = i \tan\left(\frac{w}{2}\right) \alpha\,. \tag{3.10}$$

From the conditions that the complex structure is integrable (2.5) we determine a functional relation between a and g and the expressions for $\mathcal{A}$ and $\mathcal{B}$

$$\begin{aligned} \frac{e^{2g}+1+a^2}{2a} &= -\cosh t\,, \\ \mathcal{A} &= \frac{\cosh t + a}{\sinh t}\,, \\ \mathcal{B} &= \frac{e^g}{\sinh t}\,. \end{aligned} \tag{3.11}$$

From the scalar, vector and tensor conditions (2.4)–(2.7) we obtain a pair of coupled first-order differential equations for the quantities a and $v = e^{6p+2x}$

$$\begin{aligned} a' &= -\frac{\sqrt{-1-a^2-2\,a\,\cosh t}\;(1+a\,\cosh t)}{v\,\sinh t} - \frac{a\,\sinh t\;(t+a\,\sinh t)}{t\,\cosh t-\sinh t}\,, \\ v' &= \frac{-3\,a\,\sinh t}{\sqrt{-1-a^2-2\,a\,\cosh t}} + \\ &\quad +v\,[-a^2\cosh^3 t + 2\,a\,t\coth t + a\cosh^2 t\,(2-4\,t\coth t) + \cosh t\,\times \\ &\qquad \times\,\big(1+2\,a^2-\big(2+a^2\big)\,t\coth t\big) + t\,\mathrm{csch}\,t]/ \\ &\quad /\,\big[\big(1+a^2+2a\cosh t\big)\,(t\cosh t-\sinh t)\big] \end{aligned} \tag{3.12}$$

and a set of algebraic and differential equations that allow to determine all the other unknown functions in terms of the quantity a,

$$\begin{aligned} b &= -\frac{t}{\sinh t}\,, \\ h_1 &= h_2\cosh t + Q, \\ h_2' &= -\frac{\big(t-a^2t+2at\cosh t+a^2\sinh 2t\big)}{(1+a^2+2a\cosh t)\,(-1+t\coth t)}h_2\,, \end{aligned}$$

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$$
\begin{aligned}
\chi' &= \frac{a h_2\,(1+a\cosh t)\,(2t-\sinh 2t)}{(1+a^2+2a\cosh t)\,(-1+t\coth t)}\,,\\
A' &= -\frac{(-1+t\coth t)\,(-\cosh t+t\operatorname{csch} t)}{8\,\sinh t}e^{-2x+2\phi}\,,\\
\sin w &= -\frac{2e^{x-\phi}\,(1+a\cosh t)}{\sqrt{-1-a^2-2a\cosh t}\;(-1+t\coth t)}\,,\\
\cos w &= \frac{2h_2\sinh t}{e^{\phi}\,(1-t\coth t)} = \eta e^{\phi}\,,
\end{aligned}
\tag{3.13}
$$

where η is an integration constant.

Even if analytically difficult or impossible to solve, equations (3.12) completely determine a and v in terms of two extra integration constants. At this point, equations (3.13) allow to determine all the other unknown functions $x, p, A, w, b, \chi, h_1, h_2$. Finally, the condition $e^A = |\alpha|^2 + |\beta|^2$ determines α and β. The supersymmetric conditions are now completely satisfied.

3.2 The MN and KS solutions

Our solution includes the MN and KS backgrounds as particular cases. They correspond to special expressions for the functions A, a , p, x, g, h_1, h_2, χ, b, $\mathcal{A}$, $\mathcal{B}$, α, β.

The MN string frame metric for D5-branes can be recovered for the following expressions for the metric functions[5]

$$
\begin{aligned}
a &= -\frac{t}{\sinh t}\,, & e^{2g} &= -1+2t\coth t-\frac{t^2}{\sinh^2 t}\,,\\
e^{2\phi} &= e^{2\phi_0}e^{-g}\,\sinh t\,, & e^{2A} &= e^{2A_0}e^{-g/2}\sqrt{\sinh t}\,,\\
e^{x} &= e^{\phi_0}e^{g/2}\frac{\sqrt{\sinh t}}{2}\,, & e^{6p} &= \frac{4\,e^{-2\phi_0}}{\sinh t}\,,
\end{aligned}
\tag{3.14}
$$

and of the fluxes

$$
h_1 = h_2 = \chi = K = 0 \qquad \text{and} \qquad a = b\,. \tag{3.15}
$$

The SU(3) structure that gives the results of table 2 is given by (3.8) for

$$
\mathcal{A} = \coth t - t\operatorname{csch}^2 t, \qquad \mathcal{B} = \operatorname{csch} t\sqrt{-1+2t\coth t - t^2\operatorname{csch}^2 t}\,. \tag{3.16}
$$

It is easy to check that the functions given above solve the susy equations (3.11)–(3.13). They correspond to a type C solution in table 2 with $\beta = i\alpha$, which corresponds to $w = \pi/2$.

[5]Our solution with these values of parameters coincides with the MN solution as given in the original paper [3] when we identify our radial variable t with $2r$ and change sign to a (which can be done with a change of coordinates). Recall also that we are writing the metric in the string frame metric as opposed to Einstein frame used by PT, and thus the functions in the metric given here look different than those in [10].

Similarly the KS solution is obtained by setting in the metric

$$\begin{aligned}
A &= -\frac{1}{4}\ln h\,, & a &= -\frac{1}{\cosh t}\,,\\
e^{6p+2x} &= \frac{3}{2}\left(\coth t - t\,\mathrm{csch}^2 t\right), & e^g &= \tanh t\,,\\
e^{2x} &= e^{2\phi_0}\frac{(\sinh t\,\cosh t - t)^{2/3}}{16}h & e^{\phi} &= e^{\phi_0}\,,
\end{aligned} \tag{3.17}$$

where

$$h' = -8\frac{(t\coth t - 1)(\frac{1}{2}\sinh(2t) - t)^{1/3}}{(\sinh t)^2}\,, \tag{3.18}$$

and in the fluxes,

$$\begin{aligned}
h_1 &= \frac{1}{2}(\coth t - t\coth^2 t)e^{\phi_0} & b &= -\frac{t}{\sinh t}\,,\\
h_2 &= -\frac{(-1 + t\coth t)}{2\sinh t}e^{\phi_0} & \chi &= 0\,.
\end{aligned} \tag{3.19}$$

Finally, the SU(3) complex structure is given by (3.8) for [6]

$$\mathcal{A} = e^g\,, \qquad \mathcal{B} = -a\,. \tag{3.20}$$

In this case the susy conditions (3.11)–(3.13) are identically satisfied with $w = 0$ and $\eta = e^{-\phi_0}$.

Metrically, the deformed conifold is an S^1 bundle labeled by the coordinate ψ in (3.3) over $S^2 \times S^2$ parameterized by (θ_1, ϕ_1) and (θ_2, ϕ_2). The metric has a $\mathbb{Z}_2$ symmetry corresponding to the exchange $(\theta_1, \phi_1) \to (\theta_2, \phi_2)$. This can be seen from the vielbeins (3.2)–(3.3) and the fact that a and g in KS are related by $e^{2g} = 1 - a^2$. Actually, the full solution including fluxes has an interchange symmetry $\mathcal{I}$ which is a combination of the $\mathbb{Z}_2$ symmetry plus a reversal of the signs of B_2 and C_2 (the $-\mathbb{I}$ of $SL(2, Z)$). The MN solution does not have such a symmetry, which is also broken in the perturbative solution which we will discuss next.

3.3 The GHK solution

A first attempt to go beyond the KS solution was done in [5], where a first order deformation of KS was constructed by solving the supergravity equations of motion. This deformation breaks the $\mathbb{Z}_2$ symmetry of KS. In terms of the PT ansatz (3.1)–(3.2) it consists of turning on an additional component in the NS three-form, χ', and of modifying the metric components a and g[7]

$$a \to a\,(1 + Z(t)q)\,, \qquad e^g \to e^g\,(1 + Z(t)q)\,, \tag{3.21}$$

where q is the expansion parameter. The $\mathbb{Z}_2$ breaking is reflected by the fact that the deformation (3.21) does not respect the relation $e^{2g} = 1 - a^2$ between a and g in KS.

[6]This choice of complex structure is related to the one used in [10] by an SU(3) rotations of the vielbeins.

[7]Our Z is related to GHK notations by $z_{GHK} = e^{-g}Z$.

The other functions are not modified at first order. The equations of motion fix the function Z, and also relate it to the deformation χ' in the NS flux:

$$\begin{aligned} Z(t) &= \frac{(-t + \tanh t)}{(-t + \cosh t \sinh t)^{1/3}}\,, \\ \chi'(t) &= -\frac{1}{2}\coth t \frac{\sinh 2t - 2t}{\sinh t^2} Z(t)\,. \end{aligned}$$

The supersymmetry of this solution was not checked in [5]. With our choice of complex structure it is possible to show that the deformation satisfies the susy conditions (3.11)–(3.13) with $\alpha = e^{A/2}, \beta = -i(q/4)e^{-\phi_0}e^{-3A/2}$.

It is interesting to examine how supersymmetry works in terms of torsions. The flux G is still ISD at first order and the metric still conformally Ricci-flat. However, the GHK solution has a non-zero W_3. This is possible because β is deformed at first order. The examination of the equations (2.4)–(2.7) shows that W_4, W_5, A and ϕ do not acquire a first order correction. The conformal relation $2W_5 = 3W_4$ is still satisfied but the metric is no longer conformally Calabi-Yau because $W_3 \neq 0$. The reason for Ricci-flatness is that the terms in the Ricci tensor which depend on W_3 alone are quadratic, so just like the self-duality of G, the Ricci-flatness of the metric will be violated at second order.

4. A family of IIB backgrounds

We wish to go beyond the linear order perturbation of GHK, and find a one parameter family of exact IIB backgrounds that starts from KS and goes up to MN solution, passing through GHK in the vicinity of KS.

The susy conditions (3.11)–(3.13) are suitable for a perturbative expansion in the deformation parameter q. For example, we can explicitly solve them at second order in q finding a regular solution. At this order a perturbation for the functions A, p, x, h_1, h_2, b is turned on. The dilaton also starts to run. The expression for a is particularly simple

$$a \to a(1 + qZ + q^2Z^2)\,. \tag{4.1}$$

It is unlikely that this simplicity persists at third order.

The perturbative expansion in q is the best we can do analytically. However, the existence of a regular second order solution suggests that there is indeed a one parameter family of KS deformations. This expectation can be confirmed by a numerical analysis. In the following, we study the IR and UV asymptotics for the solution and provide numerical interpolations. Here we anticipate the main results.

There is a family of regular solutions that can be parameterized by the constant appearing in the IR expansion for a

$$a = -1 + \xi t^2 + O(t^4)\,. \tag{4.2}$$

ξ ranges in the interval $[1/6, 5/6]$, with $\xi = 1/2$ corresponding to KS. The range $[1/2, 5/6]$ is related to $[1/6, 1/2]$ by the $\mathbb{Z}_2$ symmetry. All the arbitrary constants in the supersymmetry equations (except for one arbitrary additive constant in the dilaton) are fixed in terms of ξ

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by requiring IR regularity and the absence of an asymptotically flat region in the UV. For all values $1/6 < \xi \leq 1/2$ the solution is asymptotic in the UV to the KS solution and the dilaton is bounded. By fixing the value of the dilaton at $t = 0$, we can find a flow between KS and MN. Indeed, for $\xi \to 1/6$ the asymptotic suddenly changes, the dilaton blows up in the UV and the solution smoothly approaches MN.

4.1 Numerical Analysis of the Family of Solutions

As already mentioned, we were not able to find analytical solutions to the system of equations (3.12). However it is possible to show that a one parameter family of regular solutions exists by performing a power series expansion near $t = 0$. In this Section we discuss some details of this analysis.

Let us consider first the IR behavior of the solution. We calculated the first fourteen terms of the expansion around $t = 0$ for a and v; the results up to order t^4 are

$$a = -1 + \xi\, t^2 + \frac{\left(-3 + 29\,\xi - 114\,\xi^2 + 36\,\xi^3\right)\, t^4}{60} + O(t^6)\,, \tag{4.3}$$

$$v = t + \frac{\left(5 - 84\,\xi + 84\,\xi^2\right)\, t^3}{120} + O(t^5)\,. \tag{4.4}$$

We could expect a two-parameter space of solutions of the system (3.12). However, one can see that for one of the two parameters the solutions are not regular in $t = 0$.[8] Notice also that $a(0) = -1$ and $v(0) = 0$ for any regular solution. ξ parameterizes the family of solutions of the system (3.12), with $\xi = 1/2$ and $\xi = 1/6$ corresponding to the KS and MN solutions, respectively. A numerical analysis shows that the solutions exist and are regular for $1/6 \leq \xi \leq 1/2$. Moreover for all the solutions in this range $a \to 0$ for $t \to \infty$. Actually we will see in the next section that, because of a $\mathbb{Z}_2$ symmetry around $\xi = 1/2$, the flow of solutions exists for $1/6 \leq \xi \leq 5/6$. Outside this range of values we did not find (with numerical simulations) a regular solution surviving for every $t > 0$.

Knowing the series expansions for a and v, we can use the conditions (3.13) to determine the other unknown functions. We list for every quantity the first two non-zero terms in the resulting series:

$$\phi = \phi_0 + \frac{(1-2\,\xi)^2\, t^2}{4} + \frac{(1-2\,\xi)^2\, \left(13 - 132\,\xi + 132\,\xi^2\right)\, t^4}{480} + O(t^6)\,, \tag{4.5}$$

$$e^x = \frac{e^{\phi_0}\,\lambda\, t}{2} + e^{\phi_0}\,\frac{\left(-40 + 3\,\left(35 - 108\,\xi + 108\,\xi^2\right)\,\lambda^2\right)\, t^3}{720\,\lambda} + O(t^5)\,, \tag{4.6}$$

$$h_2 = e^{\phi_0}\sqrt{4 - 9\,(1-2\,\xi)^2\,\lambda^2}\left[\frac{-t}{12} + \frac{\left(-2 + 15\,\xi - 15\,\xi^2\right)\, t^3}{90} + O(t^5)\right], \tag{4.7}$$

$$\sin w = \frac{(3 - 6\,\xi)\,\lambda}{2} - \frac{(-1+2\,\xi)\,\left(-4 + 9\,(1-2\,\xi)^2\,\lambda^2\right)\, t^2}{24\,\lambda} + O(t^4)\,, \tag{4.8}$$

$$A = A_0 + \frac{t^2}{18\,\lambda^2} + \frac{\left(40 + \left(-51 - 36\,\xi + 36\,\xi^2\right)\,\lambda^2\right)\, t^4}{6480\,\lambda^4} + O(t^6)\,. \tag{4.9}$$

The expressions for b, g, h_1, χ' can be obtained by the algebraic rules (3.11),(3.13).

[8] See also in ref. [5, eq. (5.75)].

The IR solution depends on three independent integration constants: ξ, ϕ_0 and λ. Indeed the constant A_0 can be reabsorbed in a rescaling of the space time coordinates x_m in $e^{2A}dx_m^2$. The constant ϕ_0 can take any real value and corresponds to the value of the dilaton in $t=0$. The third parameter, λ, describes the behavior of e^x near $t=0$ and it determines the radius of the IR S^3. Its role in the gauge theory is explained in Section 5. $\lambda \sim 0.93266$ for KS and $\lambda = 1$ for MN.[9] Notice also that the quantities a, v, b, g do not depend on ϕ_0 and λ, and h_2 depends on these parameters only through an overall multiplicative factor.

We now study the susy equations near $t \to \infty$. This can be done by expanding all functions in our ansatz in powers of $e^{-t/3}$ times polynomial coefficients in t [24]. The results for the system (a, v) are:

$$\begin{aligned} a &= -2e^{-t} + a_{UV}\,(-1+t)\,e^{-\frac{5\,t}{3}} - \frac{1}{2}\,{a_{UV}}^2\,(-1+t)^2\,e^{-\frac{7\,t}{3}} + O(e^{-3t})\,, \qquad (4.10)\\ v &= \frac{3}{2} + \frac{9}{16}\,{a_{UV}}^2\,e^{-\frac{4\,t}{3}}\,\left(6-4\,t+t^2\right) - \frac{3}{32}\,e^{-2\,t}\,\left(33\,{a_{UV}}^3 + 16\,\left(7-3\,c+4\,t\right)\right) + O(e^{-\frac{8\,t}{3}})\,. \end{aligned}$$

The UV behaviors are parametrized by a_{UV}. The second integration constant for a and v, called c,[10] can be considered as a function of a_{UV}: $c = c(a_{UV})$. Indeed we know that there is only a one parameter family of solutions regular in $t=0$. The parameter a_{UV} is the ultra-violet analogue of ξ: it drives the solutions from KS to MN, corresponding to $a_{UV}=0$ and $a_{UV}=-\infty$, respectively.[11]

The first terms in the large t expansion of the other quantities read

$$\begin{aligned} \phi &= \phi_{UV} + \frac{3}{64}\,{a_{UV}}^2\,e^{-\frac{4\,t}{3}}\,(1-4\,t) + O\left(e^{-\frac{8\,t}{3}}\right), \qquad (4.11)\\ h_2 &= d\left[e^{-t}\,(1-t) + \frac{3}{32}\,{a_{UV}}^2\,e^{-\frac{7\,t}{3}}\,(-1+t)\,(-1+4\,t) + O(e^{-3\,t})\right], \qquad (4.12)\\ e^{2x} &= \frac{e^{2\,\phi_{UV}} - d^2}{{a_{UV}}^2}\,e^{\frac{4\,t}{3}} + \qquad (4.13)\\ &\quad + \frac{1}{32}\left(-2\,d^2\,(5-2\,t)^2 + e^{2\,\phi_{UV}}\,\left(47-28\,t+8\,t^2\right)\right) + O(e^{-\frac{2\,t}{3}})\,,\\ \cos w &= d\,e^{-\phi_{UV}}\left[1 - \frac{3}{64}\,{a_{UV}}^2\,e^{-\frac{4\,t}{3}}\,(-1+4\,t)\right] + O\left(e^{-\frac{8\,t}{3}}\right), \qquad (4.14)\\ e^{2A} &= e^{2A_1}\left[1 - \frac{3{a_{UV}}^2 e^{2\,\phi_{UV}}}{64\,(e^{2\,\phi_{UV}} - d^2)}(4t-1)e^{-\frac{4\,t}{3}} + O\left(e^{-\frac{8\,t}{3}}\right)\right]. \qquad (4.15) \end{aligned}$$

The other two UV integration constants are ϕ_{UV} and d. The first is the UV value of the dilaton. If we fix ϕ_0, $\phi_{UV}(\xi)$ is a function of the flow parameter ξ, which can be found by numerical analysis. Obviously $\phi_{UV}(1/2) = \phi_0$ in the KS case; numerical

[9]The range of λ should be determined by imposing that $|\sin w|$ and $|\cos w|$ are less than 1 for every t and that the exponentials e^x or e^p are always positive.

[10]We find it for the first time in v in the coefficient of $e^{-2\,t}$ and in a in the coefficient of $e^{-\frac{11\,t}{3}}$

[11]In fact (4.10) does not reproduce the ultraviolet behavior for the MN solution, which for a reads $a = -2te^{-t} + O(e^{-3\,t})$. The system (3.12) admits another solution in power series for large t which corresponds to MN. This second solution does not contain any arbitrary integration constant.

estimates of $\phi_{UV}(\xi)$ indicate that it diverges approaching MN. The second constant d appears multiplicatively in the equation for h_2: it is the ultraviolet analogue of λ, or better of a combination of λ and ξ.

The UV integration constants can be given in terms of the IR ones by matching the solution from the regions near $t = 0$ and near $t \to \infty$. One can for instance find the relation between a_{UV} and ξ. This can be done numerically and it shows that $a_{UV} \to -\infty$ when $\xi \to 1/6$. The interval $1/6 \leq \xi \leq 1/2$ gets mapped into $-\infty < a_{UV} \leq 0$ and this also shows that for $\xi < 1/6$ one should not expect regular solutions approaching zero for $t \to \infty$ as in (4.10). a_{UV} can also assume positive values, which correspond to $1/2 \leq \xi \leq 5/6$, since the $\mathbb{Z}_2$ symmetry acts as $a_{UV} \to -a_{UV}$ (see next section). Similarly one can determine the behavior of $d(\xi, \lambda)$.

In fact we can do more and deduce an exact relation between the IR and UV parameters. We know that $e^{\phi}/\cos w$ is a constant for every value of t (see (3.13)). Thus equating the two constants obtained by expanding the expression $e^{\phi}/\cos w$ for small t and for large t we get:

$$d\, e^{-2\,\phi_{UV}} = e^{-\phi_0} \sqrt{1 - \frac{9\,(1-2\,\xi)^2\,\lambda^2}{4}}\,. \tag{4.16}$$

In the notations of equation (3.13), $\eta = d\, e^{-2\phi_{UV}}$.

At this level we still have a solution labeled by two independent parameters (plus the parameter for the dilaton). However, for generic values of d and ϕ_{UV}, e^{2x} diverges exponentially and, moreover, e^{2A} approaches a constant for $t \to \infty$. In order to eliminate the asymptotic minkowskian region we fix $d = e^{\phi_{UV}}$. This is a requirement from AdS/CFT correspondence, since we want our supergravity solution to have a boundary for $t \to \infty$. The UV expansion is now, up to an arbitrary multiplicative constant:

$$e^{2A} = \frac{e^{\frac{2\,t}{3}}}{\sqrt{-1+4\,t}} + \frac{{a_{UV}}^2\,\left(-847 + 1752\,t - 864\,t^2 + 256\,t^3\right)}{1024\,(-1+4\,t)^{\frac{3}{2}}} e^{-\frac{2\,t}{3}} + O\left(e^{-\frac{4\,t}{3}}\right). \tag{4.17}$$

The expression (4.16), for $d = e^{\phi_{UV}}$, allows also to determine λ in terms of ξ. For $\xi \to 1/6$ or $\xi \to 5/6$, $e^{\phi_{UV}}$ diverges. This implies that λ approaches 1, which is the value for MN.

In figure 1 we plot the behavior of the functions a, ϕ, h_2 and e^x as a function of t for several values of ξ and in figure 2 we plot the behavior of the UV values a_{UV} and ϕ_{UV} as a function of ξ. In these plots we have fixed ϕ_0. Notice that the dilaton is always bounded except for the values $\xi \to 1/6$ or $\xi \to 5/6$ where it diverges at large t.

4.2 The $\mathbb{Z}_2$ symmetry

As anticipated in the previous section, our susy equations have a $\mathbb{Z}_2$ symmetry: for any interpolating solution (a, v) of the system (3.12), we have another solution $(\tilde{a}, v)$ with the same function $v \equiv e^{2x+6p}$, but, in general, with a different form for a. In this Section we will show that $\mathbb{Z}_2$ acts on the flow parameter ξ as $\xi \to (1-\xi)$ and on the UV parameter a_{UV} as $a_{UV} \to -a_{UV}$. Note that the KS solution $\xi = 1/2$ or $a_{UV} = 0$ is invariant under $\mathbb{Z}_2$, while as soon as we move from KS along the flow of solutions, the $\mathbb{Z}_2$ symmetry is broken.

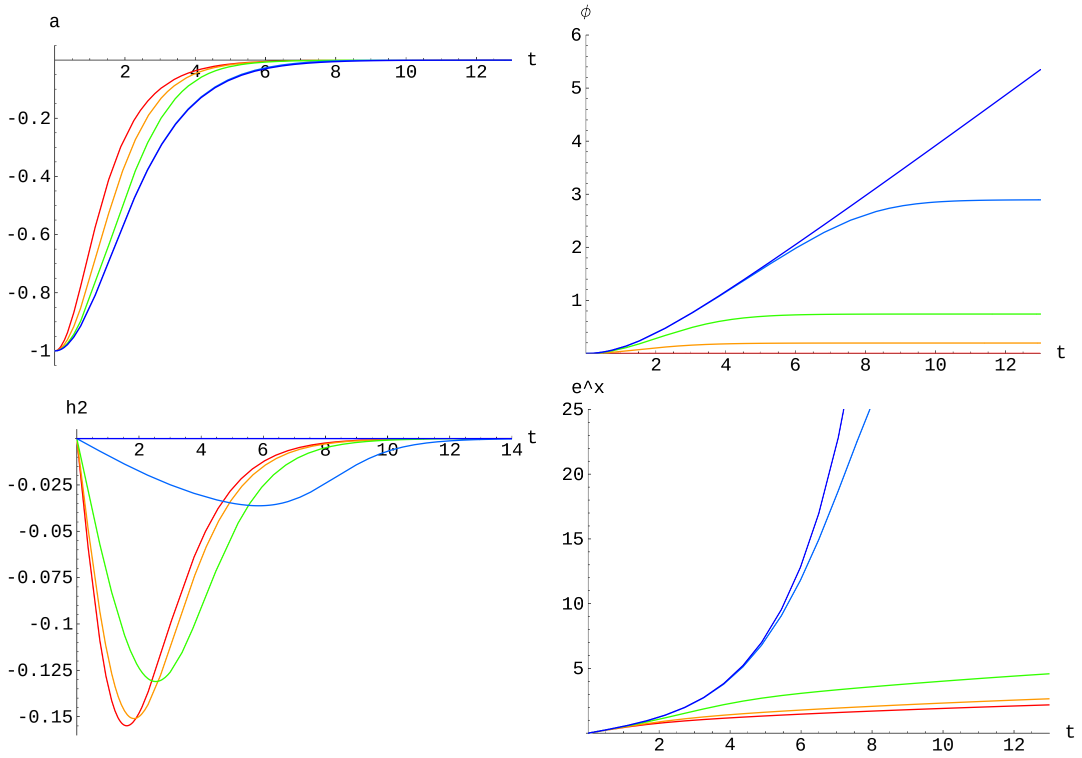

Figure 1: Graphics of the functions a, ϕ, h_2 and e^x for different values of the flow parameter ξ: the red curves correspond to KS solution ($\xi = 1/2$), the blue curves to MN solution ($\xi = 1/6$) and the orange, green and light blue curves correspond respectively to the interpolating values $\xi = 0.3$, $\xi = 0.2$ and $\xi = 0.167$.

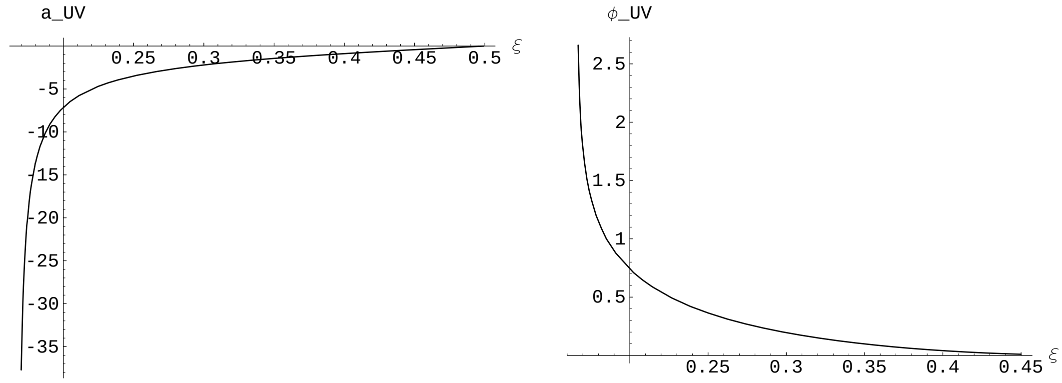

Figure 2: Graphics of a_{UV} and ϕ as a function of the flow parameter ξ, for the fundamental region $1/6 \le \xi \le 1/2$. KS solution corresponds $\xi = 1/2$ while MN to $\xi = 1/6$.

To prove such a symmetry one can introduce the following expression

$$m \equiv a\, e^{-g} = \frac{a}{\sqrt{-1 - a^2 - 2\, a\, \cosh t}}\,, \tag{4.18}$$

where we used the explicit algebraic relation between g and a. The system (3.12) can be unambiguously rewritten in terms of the new variables (m,v):

$$\begin{aligned} m' &= \frac{-1-m^2+m^2\cosh^2 t}{v\sinh t}+\frac{m\sinh t\left(t+m^2 t-m^2\cosh t\sinh t\right)}{-t\cosh t+\sinh t}\,, && (4.19)\\ v' &= \frac{m^2\cosh^3 t-\cosh t\left(-1+m^2+\left(2+m^2\right)\,t\coth t\right)+\left(1+m^2\right)\,t\,\mathrm{csch}\,t}{-\sinh t+t\cosh t}\,v-\\ &\quad -3\,m\sinh t\,. && (4.20)\end{aligned}$$

Inverting the relation (4.18) between a and m we find a second order equation with the following solutions:

$$a_\pm = \frac{-m^2\cosh t\pm\sqrt{-m^2-m^4+m^4\cosh^2 t}}{1+m^2}\,. \tag{4.21}$$

Therefore given a solution (m,v) for the system (4.19), (4.20), we obtain two solutions for (3.12) with the same function v: (a_+,v) and (a_-,v). These are the two solutions connected by the $\mathbb{Z}_2$ symmetry. A power series solution around $t=0$ of (4.19), (4.20) shows that $a_+\sim -1+\xi t^2$ and $a_-\sim -1+(1-\xi)t^2$. We see therefore that the $\mathbb{Z}_2$ action is $\xi\to(1-\xi)$. Similarly one can see that under $\mathbb{Z}_2$ $a_{UV}\to -a_{UV}$.[12]

So we have shown that the combination $m=a\,e^{-g}$ is invariant under $\mathbb{Z}_2$. In the same way, replacing a with m in the equations for ϕ, h_2, x, p and A it is easy to prove that also these quantities are $\mathbb{Z}_2$ invariant. On the contrary $\chi'\to-\chi'$.

Since we know the exact form for a in the MN case $\xi=1/6$, one can deduce from (4.18), (4.21) the exact solution when $\xi=5/6$: we find $a=t/(-2t\cosh t+\sinh t)$. Note also that while in the MN solution the angle w is equal to $\pi/2$, its $\mathbb{Z}_2$ symmetric solution has $w=-\pi/2$.

Now we want to relate the $\mathbb{Z}_2$ symmetry of the susy equations to the $\mathbb{Z}_2$ symmetry of metric[5] corresponding to the exchange of the two spheres $(\theta_1,\phi_1)\leftrightarrow(\theta_2,\phi_2)$. The PT metric ansatz can be written as

$$\begin{aligned} ds^2 &= e^{2A}dx_m^2+e^x(e^g+a^2e^{-g})(e_1^2+e_2^2)+e^{x-g}(\epsilon_1^2+\epsilon_2^2)-2ae^{x-g}(e_1\epsilon_1+e_2\epsilon_2)+\\ &\quad +e^{-6p-x}(\tilde{\epsilon_3}^2+dt^2)\,. \end{aligned} \tag{4.22}$$

It is easy to check from (4.18), (4.21) that under $\mathbb{Z}_2$: $e^g+a^2e^{-g}\leftrightarrow e^{-g}$ thus exchanging the coefficients in front of the two S_2. In the same way one can check that the fields transform under $\mathbb{Z}_2$ as: $F_3\to -F_3$, $H_3\to -H_3$, $F_5\to F_5$. This is just the action described in [5].

Since the $\mathbb{Z}_2$ symmetry is implemented by a change of coordinates, two vacua related by a $\mathbb{Z}_2$ transformation are equivalent.

5. Gauge theory

The gauge theory dual to the supergravity background corresponding to the KS solution was studied in [2], where it was identified as the gauge theory on a stack of N regular and

[12] It is also possible to show that the constant c in the expansion for v has the form $c=3+11/16\,a_{UV}^3+f_p(a_{UV})$, where f_p is an even function $f_p(a_{UV})=f_p(-a_{UV})$ and $f_p(0)=0$ (the coefficient 3 is determined by a comparison with the first order GHK deformation).

M fractional D3-brane at the apex of a conifold. The resulting theory is a $\mathcal{N}=1$ susy $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ gauge theory with chiral fields A_i, B_j respectively in the $(N+M,\overline{N})$ and $(\overline{N+M},N)$ of the gauge group, transforming as doublets of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ global symmetry group. The theory undergoes repeated Seiberg-duality transformations in which $N\to N-M$, until in the far infrared the gauge group is reduced to $\mathrm{SU}(M+p)\times\mathrm{SU}(p)$, with $0\leq p<M$. The supergravity background in [2], like the interpolating backgrounds in this paper, corresponds to $p=0$ since the field F_5 approaches zero for $t=0$; in this case the supergravity solution is regular in the IR and reliable for large values of the 't Hooft coupling g_sM; the factor e^{2A} is constant for $t=0$ and consequently the gauge theory is confining, as shown in [2]; moreover it has other interesting features such as chiral symmetry breaking $\mathbb{Z}_{2M}\to\mathbb{Z}_2$ via gluino condensate, domain walls, magnetic screening.

When $p=0$ the gauge group in the far IR is $\mathrm{SU}(M)$, and as suggested in [4, 2], the last step in the duality cascade $(\mathrm{SU}(2M)\times\mathrm{SU}(M))$ is on the baryonic branch, i.e. the $U(1)_B$ global symmetry ($A_i\to e^{i\alpha}A_i$, $B_j\to e^{-i\alpha}B_j$) is broken by the expectation values of baryonic operators:

$$\mathcal{B}\sim\epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}}(A_1)_1^{\alpha_1}(A_1)_2^{\alpha_2}\ldots(A_1)_M^{\alpha_M}(A_2)_1^{\alpha_{M+1}}(A_2)_2^{\alpha_{M+2}}\ldots(A_2)_M^{\alpha_{2M}} \tag{5.1}$$

$$\bar{\mathcal{B}}\sim\epsilon^{\alpha_1\alpha_2\ldots\alpha_{2M}}(B_1)^1_{\alpha_1}(B_1)^2_{\alpha_2}\ldots(B_1)^M_{\alpha_M}(B_2)^1_{\alpha_{M+1}}(B_2)^2_{\alpha_{M+2}}\ldots(B_2)^M_{\alpha_{2M}} \tag{5.2}$$

The baryonic branch has complex dimension 1, and it can be parametrized by ζ:

$$\mathcal{B}=i\zeta\Lambda_{2M}^{2M},\qquad\bar{\mathcal{B}}=\frac{i}{\zeta}\Lambda_{2M}^{2M}\,, \tag{5.3}$$

where Λ_{2M} is the UV scale of the gauge group $\mathrm{SU}(2M)$. Note that the $U(1)_B$ corresponds to changing ζ by a phase. The Goldstone boson associated to the spontaneous breaking of the $U(1)_B$ symmetry was identified in the supergravity dual as a massless pseudo-scalar bound state (glueball) in [5], where it was also suggested that D1-branes in string theory are dual to axionic strings in gauge theory that create a monodromy for this massless axion field.

By supersymmetry the Goldstone boson is in a $\mathcal{N}=1$ chiral multiplet; hence there will be a massless scalar mode, the "saxion" that must correspond to changing ζ by a positive real factor. This is a modulus of the theory whose expectation value induces a one parameter family of supersymmetric deformations. In the same paper [5], the supergravity dual of such deformations was suggested to be the GHK solution, constructed to first order in the flow parameter. We have shown that such deformation is supersymmetric (since it satisfies the susy equations we wrote), and moreover that a supersymmetric extension of it exists to all orders.

In [5], the saxion operator in the gauge theory was identified as:

$$\mathrm{Re}\,\mathrm{tr}\,[a_i^*\Box a_i-b_i^*\Box b_i]+\text{fermion bilinears}\,. \tag{5.4}$$

Where a_i and b_i are the lowest components of the chiral fields A_i and B_i. This operator is odd under the $\mathbb{Z}_2$ symmetry discussed in the previous section, which corresponds in field theory to the interchange of A_1, A_2 with B_1, B_2 (see [5]) accompanied by the charge

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coniugation. The KS solution, being invariant under $\mathbb{Z}_2$, corresponds to a vacuum where $|\mathcal{B}| = |\bar{\mathcal{B}}| = \Lambda_{2M}^{2M}$, and giving a non zero vev to the saxion operator (5.4), that is perturbing $|\zeta|$, should be the gauge theory analogue of our $\mathbb{Z}_2$ breaking supergravity solution.

A word of clarification is due about the uses of "interpolation" and "baryonic branch". The solution constructed in this paper depends on two parameters, ξ and the additive constant in the dilaton. In the previous Section we constructed an interpolation between KS and MN (type B and C). This interpolation involves fixing the IR value of the dilaton ϕ_0. This necessarily implies that the UV value of the dilaton ϕ_{UV} varies along the flow. On the other hand, when discussing the dual gauge theory one has to fix the UV value of the dilaton.[13] What one calls the baryonic branch corresponds to varying ξ while keeping ϕ_{UV} fixed; otherwise in addition to changing baryonic VEVs we would be changing the parameters of the lagrangian also.

We therefore suggest that the family of deformations we have found, when ϕ_{UV} is fixed, describes different vacua of the same gauge theory. In fact, all the interpolating solutions have the same leading behavior for large values of t (with the exception of the extremal points $\xi = 1/6$ and $\xi = 5/6$): using the asymptotic expansions given in section 4.1, one can write the leading contribution to the metric:

$$ds^2 \sim \frac{\text{const.}}{L^2} \frac{r^2}{\sqrt{\log r/r_0}} dx_m^2 + L^2 \frac{dr^2}{r^2} \sqrt{\log \frac{r}{r_0}} + L^2 \sqrt{\log \frac{r}{r_0}}\, ds^2_{T^{1,1}} \,, \tag{5.5}$$

where we have defined (to leading order) $r/r_0 \sim e^{t/3}$, and $ds^2_{T^{1,1}}$ is the standard metric on $T^{1,1}$:

$$ds^2_{T^{1,1}} = \frac{1}{6} \left[(e_1^2 + e_2^2) + (\epsilon_1^2 + \epsilon_2^2) \right] + \frac{1}{9} \tilde{\epsilon}_3{}^2 \,. \tag{5.6}$$

The metric (5.5), apart from the logarithmic terms, is similar to the $AdS_5 \times T^{1,1}$ metric with radius L given by

$$L^2 = \frac{9}{\sqrt{2}} \frac{M\alpha'}{2} e^{\phi_{UV}} \,, \tag{5.7}$$

where the last exponential represents the string coupling g_s, and where we have reintroduced the factor $M\alpha'/2$ that was considered 1 in previous calculations. The asymptotic form of the UV metric (5.5) depends on the flow parameter ξ only by subleading corrections, suppressed by powers of r; also the RR and NS fields approach the asymptotic form of the KS solution. According to the standard philosophy of the AdS/CFT correspondence [25] we are lead to interpret our solution as a continuous family of vacua of the $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ theory.

The corrections to the asymptotic KS metric and fluxes are interpreted as the signal that expectation values of suitable operators are turned on. In the limit where $M \ll N$ we can be more precise. The gauge theory can be then considered as approximatively conformal and we can use the field-operator identification valid for the conformal theory associated with $AdS_5 \times T^{1,1}$ [26, 27]. Using the effective potential in [10, (5.15)] and expanding around the AdS vacuum ($P \to 0$) we can determine the masses of all the fields

[13] We thank A. Dymarsky and I. Klebanov for pointing this out.

appearing in the PT ansatz in the conformal limit [29]. We get the following values for the mass squared (in units where the AdS radius $R = 1$): $m^2 = -4, -3, -3, 0, 0, 12, 21, 32$, which, using the usual relation $\Delta = 2 + \sqrt{4+m^2}$, corresponds to a set of operators with dimension $\Delta = 2, 3, 3, 4, 4, 6, 7, 8$. The two marginal operators are associated with ϕ and h_1 that determine the coupling constants of the two groups. Using the complete classifications of KK modes on $T^{1,1}$ [27] we can tentatively identify the remaining fields as operators in the following multiplets [28, 29]:

$$\begin{array}{llll} x,p \to \mathrm{Tr}W^2\bar{W}^2 & \Delta = 6,8\,; & h_2, b \to \mathrm{Tr}W_1^2 + W_2^2\,, \mathrm{Tr}(A\bar{A}+B\bar{B})W^2 & \Delta = 3,7\,; \\ a \to \mathrm{Tr}W_1^2 - W_2^2 & \Delta = 3\,; & g \to \mathrm{Tr}A\bar{A} - B\bar{B} & \Delta = 2\,. \end{array} \tag{5.8}$$

where W_i are the superfields in the vector multiplets for the two gauge groups, and W is a combination of them (sum or difference). g in particular is associated to the lowest component of the baryonic supercurrent [25]. KS solution has a non-zero vev for all of these operators (for instance the vev for h_2 corresponds to a gaugino condensate [30]) except for the one corresponding to g which is $\mathbb{Z}_2$ odd. Our family of deformations turns on a vev for this operator too, and it possibly modifies the ones that were already there in KS. All the operators appearing in (5.8) are $\mathrm{SU}(2) \times \mathrm{SU}(2)$ invariant thus reflecting the symmetry of the baryonic branch.

The running of the couplings for the two groups is determined by the r dependence of the two functions $e^{\phi} \sim 1/g_1^2 + 1/g_2^2$ and $h_1 \sim 1/g_1^2 - 1/g_2^2$. In the near conformal limit, the logarithmic running of the couplings found in [2] is only modified at subleading order

$$\begin{aligned} \frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} &= 6M \log\left(\frac{r}{r_0}\right) + O\left(\frac{1}{r^4}\right) \\ \frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} &= \mathrm{const} + O\left(\frac{1}{r^4}\right). \end{aligned} \tag{5.9}$$

The leading order in r matches the exact NSVZ formula for the β-function of supersymmetric gauge theories [2]. The extra terms, with the AdS-inspired identification $r/r_s = \mu/\Lambda$, get the interpretation of non-perturbative corrections.

In the far infrared the behavior of the metric is always similar to the KS case: it reduces to the minkowskian space-time times an S^3, as expected for a deformed conifold. But now the radius of the three spheres depends on the flow parameter: using the small t expansions given in section 4.1 we obtain for the metric and fields:

$$ds^2 \sim e^{2A_0} dx_m^2 + e^{\phi_0} M\alpha'\,\lambda \left(d\Omega_{S^3}^2 + \frac{1}{4}\,dt^2\right) + \frac{e^{\phi_0}\lambda}{2}\,\frac{M\alpha'}{2} t^2 (e_1^2 + e_2^2) \tag{5.10}$$

$$F_3 \sim -\frac{1}{2}\left(\frac{M\alpha'}{2}\right) \tilde{\epsilon}_1 \wedge \tilde{\epsilon}_2 \wedge \tilde{\epsilon}_3 \qquad\qquad H_3 \sim 0\,, \tag{5.11}$$

where the S^2 sphere $e_1^2 + e_2^2$ shrinks to zero and there are M units of RR flux through the S^3; $d\Omega_{S^3}^2$ is the metric of a round S^3 with unit radius:

$$d\Omega_{S^3}^2 = \frac{1}{4}\left(\tilde{\epsilon}_1^2 + \tilde{\epsilon}_2^2 + \tilde{\epsilon}_3^2\right). \tag{5.12}$$

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Fixing ϕ_{UV} and varying ξ, we can describe the entire baryonic branch of the SU$(N+M)\times$ SU(N) gauge theory. However, as seen from figure 1, fixing ϕ_{UV} forces the IR value ϕ_0 to (minus) infinity near the end of the flow. From eq. (5.10) we see that the S^3 radius becomes small and the supergravity solution is strongly coupled. This means that we cannot trust the supergravity description for large values of the baryonic VEV. In particular, even though we found an interpolating solution between KS and MN, we cannot claim that MN is at the endpoint of the baryonic branch. What it is true is that we can connect KS and MN by varying simultaneously the baryonic VEV and a coupling constant, that is by moving both in the space of theories and in the space of vacua.

Formulas (5.10), (5.11) are identical to equation (12) in [13] (where our parameter λ corresponds to their b); in that paper it was shown that λ determines the tensions of k-strings, that are identified in supergravity with fundamental strings placed at $t=0$. One can therefore repeat the same supergravity calculation, finding for the tensions of k-strings,

$$T_k \sim \lambda \sin\psi \sqrt{1+(\lambda^2-1)\cos^2\psi}\,, \tag{5.13}$$

where ψ is the solution of the equation

$$\psi - \frac{\pi k}{M} = \frac{1-\lambda^2}{2}\sin(2\psi)\,. \tag{5.14}$$

The parameter λ varies smoothly along the flow from its value at KS ($\lambda \sim 0.93266$) to MN ($\lambda = 1$). The formula

$$\frac{T_q}{T'_q} = \frac{\sin\frac{\pi q}{M}}{\sin\frac{\pi q'}{M}} \tag{5.15}$$

is strictly valid only for MN, showing a non universality of the IR behavior.

6. Conclusions

In this paper we found a one-parameter family of supersymmetric regular deformations of the Klebanov-Strassler solution. The existence of these solutions supports an older claim that in the last step of the KS cascade the gauge theory is in the baryonic branch. KS solution corresponds to a particular $\mathbb{Z}_2$ symmetric point in this baryonic branch, while our family describes the whole moduli space. The solution becomes strongly coupled only for large values of the VEV. This is the only known example where the moduli space of the gauge theory is described by a family of regular supergravity solutions. Moreover, by varying also the string coupling constant (that is by moving also in the space of theories) we can smoothly connect the KS and MN solutions. It would be interesting to understand what is the physical meaning of this intepolating flow.

To find these solutions we made use of two tools: the interpolating ansatz for the metric and the fluxes due to Papadopoulos and Tseytlin [10] and the supersymmetry conditions obtained in [1]. Let us make a few comments about the ansatz, the method, and the solutions.

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As always, the first order supersymmetry equations are easier to solve than the equations of motion, and that is what we are doing here. From the other side, as it is well known, supersymmetry by itself does not guarantee a solution, and Bianchi identities plus the equations of motion for the fluxes have to be imposed. Typically it is difficult to find solutions of the supersymmetry conditions that satisfy also the Bianchi identities. In PT ansatz the fluxes are constructed in such a way that Bianchi identities are automatically satisfied and we have explicitly checked that, for our solution, the supersymmetry conditions imply the equations of motions. The use of SU(3) structures allows not simply to deal in a systematic way with the first order equations, but breaks everything into few basic representations and works representation by representation. The PT ansatz together with the SU(3) structure (or spinorial) ansatz still have 13 undetermined functions; our solution uses all of these. While the system of equations appears to be heavily overdetermined, there are many simplifications and eventually we were able to find simple analytical expressions for all the functions in terms of the solutions of two coupled differential equations. We solved these differential equations in power series for small and large radius, having thus the IR and UV asymptotics of the full solution.

Although supersymmetry did not forbid it, it was unclear that regular solutions with SU(3) structure besides KS and MN existed (as far as Polchinski-Strassler, we expect the exact solution to have SU(2) structure). Realizing that GHK has SU(3) structure is a first step. Here we see that the full one-parameter family of solutions respects the SU(3) structure of the extrema. This was a pleasant surprise, which points toward a rich structure in the space of regular $\mathcal{N} = 1$ supersymmetric solutions with fluxes. As the SU(3) structure stays intact, throughout the family there is a well-defined three form Ω without zeros such that $(d - W_5^{(\bar{3})})\Omega = 0$. Moreover we can see quite explicitly that in accordance with the general conditions for preserving $\mathcal{N} = 1$ supersymmetry we can shift W_5 away by scaling Ω and get a closed three-form (pure spinor), and thus a family of generalized Calabi-Yau manifolds.

We give the set of algebraic and differential equations governing the system. Although the full analytical solution is still missing, the power expansion for small radius proves the existence of a one parameter family of regular solutions. All arbitrary constants in the solution are fixed in terms of one integration constant (plus another constant for the dilaton) which is the parameter along the flow. The flow parameter is allowed to take values in an interval that can be split into two regions related by a $\mathbb{Z}_2$ symmetry. In the interpolating flow the KS solution corresponds to the fixed point of this symmetry, while the MN solution is attained at the extremal point of the interval. The analysis of the asymptotic UV behavior shows that the whole family behaves like KS at large radius, except at the extremal MN point, where the UV behavior changes suddenly (for example, the dilaton blows up in the UV).

From the gauge theory point of view, our family of solutions describes the baryonic branch of the confining vacua [4, 5]. Indeed, the large radius behavior of the supergravity fields suggests that our solutions have an extra non-zero VEV with respect to KS for the $\mathbb{Z}_2$ odd operator $Tr(A\bar{A} - B\bar{B})$. This behavior confirms some expectations about the IR physics of KS [4, 5]: for example they are not in the same IR universality class as the pure glue $\mathcal{N} = 1$ gauge theory, they are rather in a phase where there is confinement without mass gap.

As usual, with the supergravity solution at hand we can make qualitative and quantitative predictions on the strongly coupled regime of the dual gauge theory. Finding the supergravity dual of the pure glue $\mathcal{N} = 1$ gauge theory remains a challenge, though, which means that the study of supergravity backgrounds with fluxes is far from being exhausted.

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A. Torsion and fluxes of PT ansatz in SU(3) representations

We give the components of the torsion for the PT metric (3.1) with the SU(3) structure given by (3.8) in the rotated basis:

$$\begin{aligned}
G_1 &= E_1\,, \qquad G_2 = \mathcal{A}E_2 + \mathcal{B}E_4\,, \qquad G_5 = E_5 \\
G_3 &= E_3\,, \qquad G_4 = \mathcal{B}E_2 - \mathcal{A}E_4\,, \qquad G_6 = E_6 \qquad\qquad (A.1)\\
W_1 &= \frac{1}{6}e^{-g-3p-\frac{3x}{2}}\big(-\mathcal{B} + a^2\mathcal{B} - 2a\mathcal{A}e^g - \mathcal{B}e^{2g} - 2a\mathcal{A}e^{6p+2x} - \\
&\qquad - 2\mathcal{B}e^{g+6p+2x} - 2a'e^{6p+2x} + 2e^{g+6p+2x}(\mathcal{B}\mathcal{A}' - \mathcal{A}\mathcal{B}')\big)\,;\\
W_2 &= -\frac{2}{3}e^{-g-3p-\frac{3x}{2}}G_5\wedge G_6\big(-\mathcal{B} + a^2\mathcal{B} - 2a\mathcal{A}e^g - \mathcal{B}e^{2g} + a\mathcal{A}e^{6p+2x} + \\
&\qquad \mathcal{B}e^{g+6p+2x} + e^{6p+2x}a' - e^{g+6p+2x}(\mathcal{B}\mathcal{A}' - \mathcal{A}\mathcal{B}')\big) + \\
&\quad + e^{-g+3p+\frac{x}{2}}(G_2\wedge G_3 - G_1\wedge G_4)(a\mathcal{B} + \mathcal{A}\mathcal{B}a' + \mathcal{B}^2e^g g') - \\
&\quad - \frac{1}{3}e^{-g-3p-\frac{3x}{2}}G_1\wedge G_2(\mathcal{B} - a^2\mathcal{B} + 2a\mathcal{A}e^g + \mathcal{B}e^{2g} + 2a\mathcal{A}e^{6p+2x} - \\
&\qquad - \mathcal{B}e^{g+6p+2x} + (3\mathcal{A}^2 - 1)e^{6p+2x}a' + \\
&\qquad + e^{g+6p+2x}(\mathcal{B}\mathcal{A}' - \mathcal{A}\mathcal{B}' + 3\mathcal{A}\mathcal{B}g')) + \\
&\quad + \frac{1}{3}e^{-g-3p-\frac{3x}{2}}G_3\wedge G_4\big(-\mathcal{B} + a^2\mathcal{B} - 2a\mathcal{A}e^g - \mathcal{B}e^{2g} + 4a\mathcal{A}e^{6p+2x} \\
&\qquad + \mathcal{B}e^{g+6p+2x} + (3\mathcal{A}^2 + 1)e^{6p+2x}a' + \\
&\qquad + e^{g+6p+2x}(-\mathcal{B}\mathcal{A}' + \mathcal{A}\mathcal{B}' + 3\mathcal{A}\mathcal{B}g')\big) \qquad\qquad (A.2)
\end{aligned}$$

$$
\begin{aligned}
W_3 = & -\frac{1}{4}e^{-g-3p-\frac{3x}{2}}(G_1\wedge G_3 - G_2\wedge G_4)\wedge G_6\big(-\mathcal{B}+a^2\mathcal{B}-2a\mathcal{A}e^g-\mathcal{B}e^{2g}+ \\
& \qquad +2a\mathcal{A}e^{6p+2x}+2\mathcal{B}e^{g+6p+2x}-2e^{6p+2x}a'+ \\
& \qquad +2e^{g+6p+2x}(\mathcal{B}\mathcal{A}'-\mathcal{A}\mathcal{B}')\big)+ \\
& +\frac{\mathcal{A}}{2}e^{-g-3p-\frac{3x}{2}}(G_1\wedge G_2 - G_3\wedge G_4)\wedge G_5\times \\
& \times\big(-1+a^2+e^{2g}-2\mathcal{B}e^{6p+2x}a'+2\mathcal{A}e^{g+6p+2x}g'\big)+ \\
& +\frac{1}{4}e^{-g-3p-\frac{3x}{2}}G_2\wedge G_3\wedge G_5\times \\
& \times\big(\mathcal{B}-a^2\mathcal{B}-2a\mathcal{A}e^g-3\mathcal{B}e^{2g}+2a\mathcal{A}e^{6p+2x}+ \\
& \quad +2\mathcal{B}e^{g+6p+2x}+2(\mathcal{B}^2-\mathcal{A}^2)e^{6p+2x}a'+2e^{g+6p+2x}(\mathcal{B}\mathcal{A}'-\mathcal{A}\mathcal{B}'-2\mathcal{A}\mathcal{B}g')\big)+ \\
& +\frac{1}{4}e^{-g-3p-\frac{3x}{2}}G_1\wedge G_4\wedge G_5\times \\
& \times\big(-3\mathcal{B}+3a^2\mathcal{B}-2a\mathcal{A}e^g+\mathcal{B}e^{2g}+2a\mathcal{A}e^{6p+2x}+ \\
& \quad +2\mathcal{B}e^{g+6p+2x}-2(1+2\mathcal{B}^2)e^{6p+2x}a'+2e^{g+6p+2x}(\mathcal{B}\mathcal{A}'-\mathcal{A}\mathcal{B}'+2\mathcal{A}\mathcal{B}g')\big) \qquad \text{(A.3)} \\
W_4 = & \frac{1}{2}e^{-g-3p-\frac{3x}{2}}G_5\big(-\mathcal{A}+a^2\mathcal{A}+2a\mathcal{B}e^g-\mathcal{A}e^{2g}+2e^{g+6p+2x}x'\big) \qquad \text{(A.4)} \\
W_5^{(\bar{3})} = & \frac{1}{4}e^{-g+3p+\frac{x}{2}}(G_5-iG_6)(2a\mathcal{B}-2\mathcal{A}e^g-6e^gp'+e^gx')\,. \qquad \text{(A.5)}
\end{aligned}
$$

The components of the NS flux:

$$
\begin{aligned}
H^{(1)} = & \frac{1}{6}e^{-g+3p-\frac{x}{2}}\Big((2a\mathcal{A}e^g+\mathcal{B}(1-a^2+e^{2g}))\chi'+(-2a\mathcal{A}e^g+\mathcal{B}(1+a^2-e^{2g}))h_1'- \\
& \qquad -2(e^gh_2+(-a\mathcal{B}+\mathcal{A}e^g)h_2')\Big) \\
H^{(3+\bar{3})} = & \frac{1}{2}e^{-g+3p-\frac{x}{2}}G_5\Big((-2a\mathcal{B}e^g+\mathcal{A}(1-a^2+e^{2g}))\chi'+ \\
& \qquad +(\mathcal{A}+a^2\mathcal{A}+2a\mathcal{B}e^g-\mathcal{A}e^{2g})h_1'+2(a\mathcal{A}+\mathcal{B}e^g)h_2'\Big) \\
H^{(6+\bar{6})} = & \frac{1}{4}e^{-g+3p-\frac{x}{2}}\times \\
& \times\big[-2\mathcal{A}(G_1\wedge G_2-G_3\wedge G_4)\wedge G_5\times \\
& \quad \times((-1+a^2+e^{2g})\chi'-(1+a^2+e^{2g})h_1'-2ah_2')+G_2\wedge G_3\wedge G_5\times \\
& \quad \times(2e^gh_2+(2a\mathcal{A}e^g+\mathcal{B}(-1+a^2+3e^{2g}))\chi'- \\
& \qquad -(\mathcal{B}+a^2\mathcal{B}+2a\mathcal{A}e^g+3\mathcal{B}e^{2g})h_1'-2(a\mathcal{B}+\mathcal{A}e^g)h_2')+G_1\wedge G_4\wedge G_5\times \\
& \quad \times(2e^gh_2+(2a\mathcal{A}e^g-\mathcal{B}(-3+3a^2+e^{2g}))\chi'+ \\
& \qquad +(3\mathcal{B}+3a^2\mathcal{B}-2a\mathcal{A}e^g+\mathcal{B}e^{2g})h_1'-2(-3a\mathcal{B}+\mathcal{A}e^g)h_2')- \\
& \quad -(G_1\wedge G_3-G_2\wedge G_4)\wedge G_6((2a\mathcal{A}e^g+\mathcal{B}(1-a^2+e^{2g}))\chi'+ \\
& \quad +(-2a\mathcal{A}e^g+\mathcal{B}(1+a^2-e^{2g}))h_1'+2(e^gh_2+(a\mathcal{B}-\mathcal{A}e^g)h_2'))\big]\,. \qquad \text{(A.6)}
\end{aligned}
$$

And finally the RR three-form flux components

$$
F_3^{(1)} = -\frac{iP}{6}e^{-g+3p-\frac{x}{2}}(2\mathcal{A}(a-b)e^g+\mathcal{B}(-1-a^2+2ab+e^{2g})-2e^gb') \qquad \text{(A.7)}
$$

$$
F_3^{(3+\bar{3})} = \frac{P}{2}e^{-g+3p-\frac{x}{2}}(2(a-b)\mathcal{B}e^g+\mathcal{A}(1+a^2-2ab-e^{2g}))G_6 \qquad \text{(A.8)}
$$

$$
\begin{aligned}
F_3^{(6+\bar{6})} &= \frac{P}{4} e^{-g+3p-\frac{x}{2}} \\
&\times \Big[2\mathcal{A}(1+a^2-2ab+e^{2g})(G_1\wedge G_2 - G_3\wedge G_4)\wedge G_6 + \\
&\quad + G_1\wedge G_4\wedge G_6(-2\mathcal{A}(a-b)e^g + \mathcal{B}(3+3a^2-6ab+e^{2g}) - 2e^g b') - \\
&\quad - G_1\wedge G_3\wedge G_5(2\mathcal{A}(a-b)\, e^g + \mathcal{B}(-1-a^2+2ab+e^{2g}) + 2e^g b') + \\
&\quad + G_2\wedge G_4\wedge G_5(2\mathcal{A}(a-b)e^g + \mathcal{B}(-1-a^2+2ab+e^{2g}) + 2e^g b') - \\
&\quad - G_2\wedge G_3\wedge G_6(2\mathcal{A}(a-b)e^g + \mathcal{B}(1+a^2-2ab+3e^{2g}) + 2e^g b')\Big] . \qquad \text{(A.9)}
\end{aligned}
$$

B. Derivation of the susy equations

We derive here the susy equations for the PT ansatz imposing the conditions (2.4) to (2.7).

B.1 Conditions from W_1, W_2 and the singlets $H_3^{(1)}$, $F_3^{(1)}$

Let's start with the conditions for the integrability of the complex structure (2.5). Setting to zero all the components of W_1 and W_2 given in the appendix gives us five equations.[14] After some algebra we reduce them to a set of only two independent equations:

$$2\mathcal{A}a + \mathcal{B}e^{-g}(1-a^2+e^{2g}) = 0\,, \qquad \text{(B.1)}$$

$$a + a'\mathcal{A} + e^g g'\mathcal{B} = 0\,. \qquad \text{(B.2)}$$

Solving these for $\mathcal{A}$ and $\mathcal{B}$ and imposing the constraint $\mathcal{A}^2+\mathcal{B}^2=1$ gives an equation for g'

$$g' = e^{-2\,g}\left[a\,S + (C-a)\;a'\right]\,, \qquad \text{(B.3)}$$

where we have defined the following useful quantities:

$$S \equiv \frac{\sqrt{a^4 + 2\,a^2\,\left(-1+e^{2\,g}\right) + \left(1+e^{2\,g}\right)^2}}{2\,a}\,, \qquad \text{(B.4)}$$

$$C \equiv \frac{1+a^2+e^{2\,g}}{2\,a}\,. \qquad \text{(B.5)}$$

From these definitions it follows that $C^2-S^2=1$. Differentiating (B.5), (B.4) and using only the equation (B.3), one can show that C and S satisfy the remarkable conditions: $C'=S$ and $S'=C$, allowing us to integrate them

$$C = -\left(k_1\cosh t + k_2 \sinh t\right)\,, \qquad \text{(B.6)}$$

$$S = -\left(k_1\sinh t + k_2 \sinh t\right)\,. \qquad \text{(B.7)}$$

These are two algebraic relations that determine e^{2g} in terms of a and t. From $C^2-S^2=1$ we find the constraint: $k_1^2-k_2^2=1$.

[14] As a check of our formalism, we compared our equations with the conditions derived in [10]. The conditions derived by $W_1=W_2=0$ are equivalent to formulae [10, (4.31)] when the identifications $e^{-g_{PT}} = 2e^{-g}, t_{PT} = t/2$ and $(X,P)=(\mathcal{A},\mathcal{B})$ are made.

Parameterizing k_1 and k_2 as $k_1 = \cosh t_0$, $k_2 = \sinh t_0$, equations (B.6), (B.7) become: $C = -\cosh(t + t_0)$, $S = -\sinh(t + t_0)$. Therefore, up to a redefinition of t, we can always fix the integration parameters to $k_1 = 1$, $k_2 = 0$, corresponding to

$$C = -\cosh t\,, \qquad S = -\sinh t\,. \tag{B.8}$$

Notice that the expressions (3.14), (3.17), commonly found in the literature, are written with this choice of t and they have $0 \leq t < +\infty$. With this fixing the expression of g as a function of a (B.6) becomes:

$$e^{2g} = -1 - a^2 - 2\,a\,\cosh t\,. \tag{B.9}$$

From conditions (B.1), (B.2) and using (B.3), we can derive the following expressions for $\mathcal{A}$, $\mathcal{B}$:

$$\mathcal{A} = \frac{C - a}{S} \qquad \text{and} \qquad \mathcal{B} = -\frac{e^g}{S}\,, \tag{B.10}$$

which will be used to eliminate $\mathcal{A}$ and $\mathcal{B}$ from following formulas.

Let's turn now to the singlets conditions on $H^{(1)}$ and $F_3^{(1)}$ (2.4); they give differential equations for h_1' and b', respectively:

$$h_1' = -h_2 S - h_2' C\,, \tag{B.11}$$

$$b' = \frac{1 - b\,C}{S}\,. \tag{B.12}$$

Note that using $(B.10)$, χ' drops out of the first equation. They can both be integrated to give:

$$h_1 = -h_2 C + \tilde{Q}\,, \tag{B.13}$$

$$b = \frac{t + c_b}{S}\,, \tag{B.14}$$

where c_h and c_b are integration constants; notice that imposing regularity conditions in $t = 0$ forces us to put $c_b = 0$. With the choice $k_1 = 1$, $k_2 = 0$, $c_b = 0$, the previous expressions become:

$$h_1 = h_2 \cosh t + \tilde{Q}\,, \tag{B.15}$$

$$b = -\frac{t}{\sinh t}\,. \tag{B.16}$$

Notice in particular that b does not vary along the flow.

Summarizing, we managed to integrate the equations for g, b, h_1, so that they can always be thought as functions of a and h_2. In fact we will use their integrated expressions only in the last steps (3.12) in order to avoid the introduction of integration parameters from the beginning, and so, to simplify following formulas, we will use only their differential expressions.

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B.2 Conditions from the 6 sector

Analyzing the conditions (2.6) (note for example that the third one can be derived from the first two eliminating W_3), it is easy to show from their explicit expressions in the appendix that they give four independent equations which can be rearranged in two algebraic expressions:

$$\sin w = 2\, e^{x-g-\phi}\, \frac{a\,C-1}{b\,C-1}\,, \tag{B.17}$$

$$\cos w = 2\, e^{-\phi}\, h_2\, \frac{S}{b\,C-1}\,, \tag{B.18}$$

and two differential equations for a' and χ':

$$a' = -\frac{(a\,C-1)}{S}\, e^{g-2x-6p} + \frac{a\,(a-b)\,S}{b\,C-1}\,, \tag{B.19}$$

$$\chi' = -a\,S\, \frac{\left(b-2\,C+b\,C^2\right)\, h_2 + (b\,C-1)\, S\, h_2'}{(a\,C-1)\,(b\,C-1)}\,. \tag{B.20}$$

Notice that since the equations (2.6) are homogeneous in α and β, the algebraic expressions depend only upon w defined as in (3.10): $\beta = i\,\alpha \tan(w/2)$.

B.3 Conditions from the $\bar{3}$ sector

The strategy we used to analyze equations (2.6) and (2.7) was to write first the ratio of every expression to the first equation of (2.6) (the one for $F_3^{(\bar{3})}$), in order to simplify the dependence on $\bar{\partial}\alpha$.

The ratio between the equations for $F_5^{(\bar{3})}$ and $F_3^{(\bar{3})}$ translates into the condition for the function K in front of the PT ansatz for F_5:

$$K = h_2(C-b) \qquad \Rightarrow \qquad K = -h_1 - bh_2 + \tilde{Q}\,, \tag{B.21}$$

where in the second equality we have used the integrated expression for h_1 in order to show that this susy equation coincides with the only non trivial Bianchi identity: $dF_5 = H \wedge F$, that is $K' = -(h_1 + bh_2)'$. $\tilde{Q}$ is then identified with the quantity Q defined in the PT ansatz.

Note that the expression $h_2(C-b)$ for the five form F_5 is always vanishing at $t=0$ for all the regular interpolating solutions we discussed in section 4: inserting the small t expansion 4.7 we find $K \propto t^3$. In the language of gauge theory this should mean that the solutions found in this paper all correspond to $p=0$ where $N = nM + p$ and $0 \le p < M$, with N and M the number of regular and fractional branes respectively. This is the only value of p where a baryonic branch exists.

From the ratio of the equations $H_3^{(\bar{3})}$ and $F_3^{(\bar{3})}$ we derive

$$\chi' = -\frac{(a\,C-1)\,(b\,h_2\,S + (b\,C-1)\,h_2')}{a\,(b\,C-1)}\,. \tag{B.22}$$

Equations (B.20) and (B.22) can be rearranged in

$$h_2' = \frac{2\,a^2\,C - b\,\left(-1 + a^2 + 2\,a\,C\right)}{b\,C - 1}\,e^{-2\,g}\,h_2\,S\,, \tag{B.23}$$

$$\chi' = \frac{2\,a\,(b - C)\,(a\,C - 1)}{b\,C - 1}\,e^{-2\,g}\,h_2\,S\,, \tag{B.24}$$

and it is easy to show that equation (B.24) is equal to the equation of motion for χ.[15]

The ratios of the equations for W_4, $W_5^{(\bar{3})}$, $\bar{\partial}\phi$, $\bar{\partial}A$ with the equation for $F_3^{\bar{3}}$ give respectively the following differential equations for x', p', ϕ', A':

$$x' = a\,S\,e^{-g-6\,p-2\,x} + \frac{b - C}{b\,C - 1}\,h_2^2\,S\,e^{-2\,x}\,, \tag{B.25}$$

$$\begin{aligned} p' = &-\frac{e^{-2\,g}}{12\,S\,\left(b\,C - 1\right)} \times \\ &\times [e^{-2\,x-6\,p}(4\,\left(b - C\right)\,\left(-1 - a^2 + 2\,a\,C\right)\,e^{6\,p}\,h_2{}^2\,S^2 - 2\,a\,\left(b\,C - 1\right)\,e^{g}\,S^2) - \\ &\quad - (4a + 2b + 2a^2b - 2C - 2a^2C - 4abC + 4aS^2 + 4bS^2 + \\ &\quad\quad + 2a^2bS^2 + 2a^2CS^2 - 8abCS^2)] \end{aligned} \tag{B.26}$$

$$\phi' = \frac{\left(C - b\right)\,\left(a\,C - 1\right)^2}{\left(b\,C - 1\right)\,S}\,e^{-2\,g}\,, \tag{B.27}$$

$$A' = \frac{b - C - b^2\,C + b\,C^2}{8\,S}\,e^{-2\,x+2\,\phi}\,. \tag{B.28}$$

There are still two conditions to be imposed:

$$A = \log\left(|\alpha|^2 + |\beta|^2\right) \tag{B.29}$$

which determines the values of α and β once A and w are known, and finally the equation (2.6) for $F_3^{(\bar{3})}$. If we use the derivative of (B.29) to express α' in function of w', we get from $F_3^{(\bar{3})}$ the following differential equation for w':

$$w' = \frac{\left(b - C\right)\,\left(a\,C - 1\right)}{b\,C - 1}\,e^{-g-x}\,h_2\,. \tag{B.30}$$

B.4 Check of consistency and other useful relations

If we treat f g, b, h_1, $\mathcal{A}$, $\mathcal{B}$ as functions of a and h_2, we are left with the following unknowns: a, h_2, ϕ, x, p and w. Note in fact that A, χ', α and β are determined from the equations (B.28), (B.24), (B.29) once the other quantities are known and do not enter in other equations. So we are left with 6 unknowns and 8 equations: the two algebraic (B.17), (B.18), and the six differential (B.19), (B.23), (B.25), (B.26), (B.27), (B.30).

The system may seem overdetermined, but it is not so: it is straightforward to show that the equations for x' (B.25) and for w' (B.30) may be obtained respectively by differentiating the two algebraic equations (B.17), (B.18) and using the other equations. We

[15] This is the equation (5.21) in [10].

may therefore discard these two equations and write a set of independent equations for a, h_2, ϕ, x, $v \equiv e^{2x+6p}$ and w as:

$$\begin{aligned} a' &= -\frac{(a\,C-1)\,e^g}{S}\,\frac{1}{v}+\frac{a\,(a-b)\,S}{b\,C-1}\,, && \text{(B.31)}\\ v' &= 3ae^{-g}S+v\frac{e^{-2\,g}}{(b\,C-1)\,S}[b\left(-1+2\,a\,C+2\,C^2+a^2\,C^2-4\,a\,C^3\right)+\\ &\qquad +C\,\left(-1+2\,a\,C+a^2\,\left(-2+C^2\right)\right)]\,, && \text{(B.32)}\\ \phi' &= \frac{(C-b)\,(a\,C-1)^2}{(b\,C-1)\,S}\,e^{-2\,g}\,, && \text{(B.33)}\\ h_2' &= \frac{2\,a^2\,C-b\,\left(-1+a^2+2\,a\,C\right)}{b\,C-1}\,e^{-2\,g}\,h_2\,S\,, && \text{(B.34)}\\ \sin w &= 2\,e^{x-g-\phi}\,\frac{a\,C-1}{b\,C-1}\,, && \text{(B.35)}\\ \cos w &= 2\,e^{-\phi}\,h_2\,\frac{S}{b\,C-1}\,, && \text{(B.36)} \end{aligned}$$

where the equation for v follows directly from (B.25), (B.26). Note that the first two equations are a coupled first order system that determines a and v; the next two determine ϕ and h_2, and the algebraic ones determine w and x. Alternatively it is easy to show that one could discard the differential equation for h_2 (B.34) and keep instead that for x' (B.25). All these susy equations are satisfied for the KS, MN and GHK solutions.

We can perform a further integration for the dilaton equation: note that from (B.27), (B.30) and the algebraic relations it follows that:

$$\phi' = -w'\tan w \qquad \text{(B.37)}$$

which can be integrated to

$$e^{\phi} = \frac{\cos w}{\eta}\,, \qquad \text{(B.38)}$$

with η an integration parameter; notice that this cannot be done for the extremal MN case, since for MN $w=\pi/2$ and consequently $\eta \to 0$. The relation (B.38) determines w in terms of ϕ; we can rewrite the previous set of equations for the variables a, v, ϕ, h_2, x discarding the dilaton equation:

$$\begin{aligned} a' &= -\frac{(a\,C-1)\,e^g}{S}\,\frac{1}{v}+\frac{a\,(a-b)\,S}{b\,C-1}\,, && \text{(B.39)}\\ v' &= 3ae^{-g}S+v\frac{e^{-2\,g}}{(bC-1)\,S}[b\,\left(-1+2\,a\,C+2\,C^2+a^2\,C^2-4\,a\,C^3\right)+\\ &\qquad +C\left(-1+2\,a\,C+a^2\,\left(-2+C^2\right)\right)]\,, && \text{(B.40)}\\ h_2' &= \frac{2\,a^2\,C-b\,\left(-1+a^2+2\,a\,C\right)}{b\,C-1}\,e^{-2\,g}\,h_2\,S\,, && \text{(B.41)}\\ e^{2\phi} &= \frac{2}{\eta}\,\frac{S}{b\,C-1}\,h_2\,, && \text{(B.42)}\\ e^{2x} &= \left(\frac{e^g\,S}{a\,C-1}\right)^2\left(\frac{1}{e^{2\phi}\eta^2}-1\right)\,h_2^2\,, && \text{(B.43)} \end{aligned}$$

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where we have eliminated w from the algebraic expressions (B.17), (B.18) through (B.38) and solved with respect to ϕ and x. Notice that the relation between η and the other integration constants is:

$$\eta = d\, e^{-2\,\phi_{UV}} = e^{-\phi_0} \sqrt{1 - \frac{9\,(1-2\,\xi)^2\,\lambda^2}{4}}\,. \tag{B.44}$$

In conclusion let's say few words as one can verify that equations of motion are automatically satisfied by our susy equations. We do not report here the calculations since they are long but purely algebraic. We already said that Bianchi identities are satisfied by the PT ansatz. Then one has to write all the equations of motion for the metric, fluxes and dilaton (the equation for the axion is trivially satisfied by the PT ansatz with $C_0 = 0$) in string frame. This is a system of second order differential equations much more complicated than the susy equations for the functions A, x, p, g, a, h_1, h_2, χ, b, ϕ; obviously they do not depend on the functions from supersymmetry and complex structure α, β, w, $\mathcal{A}$, $\mathcal{B}$. Replacing every second and first derivative of the functions in the equations of motion with the derivatives of the susy differential equations (B.28), (B.25), (B.26), (B.3), (B.19), (B.11), (B.23), (B.24), (B.12), (B.27) or with the susy differential equations themselves, one finds only algebraic relations that, after simplification, can always be reduced to the identity using the susy algebraic equation that can be deduced eliminating w from (B.17), (B.18) through $\sin^2 w + \cos^2 w = 1$.

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